Physics 741 – Graduate Quantum Mechanics 1

Solutions to Chapter 11

1. [5] An electron in an unknown spin state $|a\rangle$ is brought into proximity with a second electron in a known spin state $|b\rangle$. We wish to make the spin of the second electron match the first. A *quantum Xerox device* will copy it onto the second spin, so $U_{\text{Xerox}}|a,b\rangle = |a,a\rangle$. A *quantum teleporter* will swap the two spin states, as $U_{\text{Teleport}}|a,b\rangle = |b,a\rangle$.

(a) [3] By considering the three initial spin states $|a\rangle = |+\rangle$, $|a\rangle = |-\rangle$, and $|a\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$, show that the quantum Xerox device is impossible.

If the quantum Xerox device exists, it must change the state $|+,b\rangle$ into $|+,+\rangle$ and $|-,b\rangle$ into $|-,-\rangle$, in other words

$$U_{\text{Xerox}}|+,b\rangle = |+,+\rangle \quad \text{and} \quad U_{\text{Xerox}}|-,b\rangle = |-,-\rangle$$

However, U_{Xerox} is a linear operator, and it follows that

$$U_{\text{Xerox}}\left[\tfrac{1}{\sqrt{2}}\left(|+,b\rangle + |-,b\rangle\right)\right] = \tfrac{1}{\sqrt{2}}\left(|+,+\rangle + |-,-\rangle\right).$$

However, the quantum Xerox device is supposed to evolve this state into

$$U_{\text{Xerox}}\left[\tfrac{1}{\sqrt{2}}\left(|+,b\rangle + |-,b\rangle\right)\right] = \tfrac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \otimes \tfrac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) = \tfrac{1}{2}\left(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle\right)$$

Obviously, these equations are inconsistent, and hence this is impossible.

(b) [2] By considering the same three initial states, show that the same problem does not apparently occur for the quantum teleport device.

The quantum teleport device should evolve the states according to

$$U_{\text{Teleport}}|+,b\rangle = |b,+\rangle \quad \text{and} \quad U_{\text{Teleport}}|-,b\rangle = |b,-\rangle$$

and therefore by linearity,

$$U_{\text{Teleport}}\left[\tfrac{1}{\sqrt{2}}\left(|+,b\rangle + |-,b\rangle\right)\right] = \tfrac{1}{\sqrt{2}}\left(|b,+\rangle + |b,-\rangle\right)$$

But this is exactly what we would want it to do, so there is, in fact, no problem in this case.

2. [10] At t = 0, the wave function of a free particle is given by

$$\Psi(x,t=0) = (A/\pi)^{1/4} \exp\left[-\tfrac{1}{2}A(x-x_0)^2\right]$$

Find $\Psi(x,t)$, using the propagator.

This is straightforward, though boring. The answer is

$$\begin{aligned}
\Psi(x,t) &= \int dx' K(x,t;x',0)\Psi(x',0) \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int dx' \exp\left[\frac{im(x-x')^2}{2\hbar t}\right] \exp\left[-\tfrac{1}{2}A(x'-x_0)^2\right] \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \int dx' \exp\left[\left(\frac{im}{2\hbar t}-\frac{A}{2}\right)x'^2 + \left(-\frac{imx}{\hbar t}+Ax_0\right)x' + \frac{imx^2}{2\hbar t} - \frac{Ax_0^2}{2}\right] \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi i\hbar t}} \sqrt{\frac{\pi}{A/2 - im/2\hbar t}} \exp\left[\frac{(-imx/\hbar t + Ax_0)^2}{4(A/2 - im/2\hbar t)}\right] \exp\left[\frac{imx^2}{2\hbar t} - \frac{Ax_0^2}{2}\right] \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{A\hbar ti + m}} \exp\left[\frac{A^2x_0^2\hbar^2t^2 - 2iAmxx_0\hbar t - m^2x^2}{2(A\hbar t - im)\hbar t} + \frac{imx^2 - \hbar tAx_0^2}{2\hbar t}\right] \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{A\hbar ti + m}} \exp\left[\frac{A^2x_0^2\hbar^2t^2 - 2iAmxx_0\hbar t - m^2x^2}{2(A\hbar t - im)\hbar t} + \frac{(imx^2 - \hbar tAx_0^2)(A\hbar t - im)}{2(A\hbar t - im)\hbar t}\right] \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{A\hbar ti + m}} \exp\left[\frac{Ax_0^2\hbar tim - 2iAmxx_0\hbar t + imA\hbar tx^2}{2(A\hbar t - im)\hbar t}\right] \\
&= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{m + A\hbar ti}} \exp\left[\frac{imA(x-x_0)^2}{2(A\hbar t - im)}\right] = \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{m + A\hbar ti}} \exp\left[-\frac{mA(x-x_0)^2}{2(m + A\hbar ti)}\right]
\end{aligned}$$

The final form makes it clear at least that if t = 0, the wave function matches the initial wave function. Though it is less obvious, the wave function will spread out over time.

3. [25] This problem should be worked entirely in the Heisenberg formulation of quantum mechanics. A particle lies in the one-dimensional harmonic oscillator potential, so $H = P^2/2m + \frac{1}{2}m\omega^2 X^2$.

(a) [5] Work out dX/dt and dP/dt.

According to the Heisenberg equations of motion,

$$\frac{dX}{dt} = \frac{i}{\hbar}[H, X] = \frac{i}{2m\hbar}\left[P^2, X\right] = \frac{i}{2m\hbar}\left(-i\hbar P - i\hbar P\right) = \frac{P}{m},$$

$$\frac{dP}{dt} = \frac{i}{\hbar}[H, P] = \frac{im\omega^2}{2\hbar}\left[X^2, P\right] = \frac{im\omega^2}{2\hbar}\left(i\hbar X + i\hbar X\right) = -m\omega^2 X.$$

(b) [5] Define the operators $a(t) = \sqrt{m\omega/2\hbar}\,X(t) + i\,P(t)\big/\sqrt{2\hbar m\omega}$ and its Hermitian conjugate $a^\dagger(t)$. Show that these satisfy equations $da(t)/dt \propto a(t)$ and $da^\dagger(t)/dt \propto a^\dagger(t)$, and determine the proportionality constant in each case.

$$\frac{da}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\frac{\partial X}{\partial t} + i\sqrt{\frac{1}{2\hbar m\omega}}\frac{\partial P}{\partial t} = \sqrt{\frac{m\omega}{2\hbar}}\frac{P}{m} - i\sqrt{\frac{1}{2\hbar m\omega}}m\omega^2 X$$

$$= -i\omega\left[\sqrt{\frac{m\omega}{2\hbar}}X + i\sqrt{\frac{1}{2\hbar m\omega}}P\right] = -i\omega a,$$

$$\frac{da^\dagger}{dt} = \sqrt{\frac{m\omega}{2\hbar}}\frac{\partial X}{\partial t} - i\sqrt{\frac{1}{2\hbar m\omega}}\frac{\partial P}{\partial t} = \sqrt{\frac{m\omega}{2\hbar}}\frac{P}{m} + i\sqrt{\frac{1}{2\hbar m\omega}}m\omega^2 X$$

$$= i\omega\left[\sqrt{\frac{m\omega}{2\hbar}}X - i\sqrt{\frac{1}{2\hbar m\omega}}P\right] = i\omega a^\dagger.$$

(c) [5] Solve the differential equations for $a(t)$ and $a^\dagger(t)$ in terms of $a(0)$ and $a^\dagger(0)$. As a check, confirm that the Hamiltonian $H = \hbar\omega\left[a^\dagger(t)a(t) + \frac{1}{2}\right]$, is independent of time.

The solutions of $\dot{a}(t) = -i\omega a(t)$ and $\dot{a}^\dagger(t) = i\omega a^\dagger(t)$ are respectively

$$a(t) = e^{-i\omega t}a(0) \quad \text{and} \quad a^\dagger(t) = e^{i\omega t}a^\dagger(0).$$

Plugging these into the Hamiltonian, we see that the time dependence goes away.

$$H = \hbar\omega\left[a^\dagger(t)a(t) + \tfrac{1}{2}\right] = \hbar\omega\left[e^{i\omega t}a^\dagger(0)e^{-i\omega t}a(0) + \tfrac{1}{2}\right] = \hbar\omega\left[a^\dagger(0)a(0) + \tfrac{1}{2}\right].$$

(d) [5] Rewrite *X*(*t*) and *P*(*t*) in terms of $a(t)$ and $a^\dagger(t)$, and rewrite $a(0)$ and $a^\dagger(0)$ in terms of *X*(0) and *P*(0), so that *X*(*t*) and *P*(*t*) depend only on *X*(0) and *P*(0). You may find the identities below useful.

$$X(t)=\sqrt{\hbar/2m\omega}\left[a(t)+a^\dagger(t)\right] \quad \text{and} \quad P(t)=i\sqrt{\hbar m\omega/2}\left[a^\dagger(t)-a(t)\right].$$

As a check, you should find $X(T)=X(0)$, if *T* is the classical period.

These are fairly straightforward. We start with the position operator:

$$X(t)=\sqrt{\frac{\hbar}{2m\omega}}\left[a(t)+a^\dagger(t)\right]=\sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}a(0)+e^{i\omega t}a^\dagger(0)\right]$$

$$=\sqrt{\frac{\hbar}{2m\omega}}\left[e^{-i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)+i\sqrt{\frac{1}{2\hbar m\omega}}P(0)\right)+e^{i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)-i\sqrt{\frac{1}{2\hbar m\omega}}P(0)\right)\right]$$

$$=\frac{1}{2}X(0)\left(e^{-i\omega t}+e^{i\omega t}\right)+\frac{i}{2m\omega}P(0)\left(e^{-i\omega t}-e^{i\omega t}\right)=X(0)\cos(\omega t)+\frac{P(0)}{m\omega}\sin(\omega t).$$

We now do the momentum operator in exactly the same way.

$$P(t)=i\sqrt{\frac{\hbar m\omega}{2}}\left[a^\dagger(t)-a(t)\right]=i\sqrt{\frac{\hbar m\omega}{2}}\left[e^{i\omega t}a^\dagger(0)-e^{-i\omega t}a(0)\right]$$

$$=i\sqrt{\frac{\hbar m\omega}{2}}\left[e^{i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)-i\sqrt{\frac{1}{2\hbar m\omega}}P(0)\right)-e^{-i\omega t}\left(\sqrt{\frac{m\omega}{2\hbar}}X(0)+i\sqrt{\frac{1}{2\hbar m\omega}}P(0)\right)\right]$$

$$=i\frac{m\omega}{2}X(0)\left(e^{i\omega t}-e^{-i\omega t}\right)+\frac{1}{2}P(0)\left(e^{i\omega t}+e^{-i\omega t}\right)=P(0)\cos(\omega t)-m\omega X(0)\sin(\omega t).$$

It is now obvious that if we set $T=2\pi/\omega$, the classical period, then $X(T)=X(0)$ and $P(T)=P(0)$.

(e) [5] Suppose the quantum state (which is independent of time) is chosen to be an eigenstate of $X(0)$, $X(0)|\psi\rangle=x_0|\psi\rangle$. Show that at each of the times $t=\frac{1}{4}T$, $t=\frac{1}{2}T$, $t=\frac{3}{4}T$, and $t=T$, it is an eigenstate of either $X(t)$ *or* $P(t)$, and determine its eigenvalue.

These times correspond to $\omega t=\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π respectively. We therefore have, at these times

$$X\left(\tfrac{1}{4}T\right)=P(0)/m\omega, \quad X\left(\tfrac{1}{2}T\right)=-X(0), \quad X\left(\tfrac{3}{4}T\right)=-P(0)/m\omega, \quad X(T)=X(0),$$
$$P\left(\tfrac{1}{4}T\right)=-m\omega X(0), \quad P\left(\tfrac{1}{2}T\right)=-P(0), \quad P\left(\tfrac{3}{4}T\right)=m\omega X(0), \quad P(T)=P(0).$$

From these it is easy to see that

$$P\left(\tfrac{1}{4}T\right)|\psi\rangle = -m\omega x_0|\psi\rangle, \quad X\left(\tfrac{1}{2}T\right)|\psi\rangle = -x_0|\psi\rangle,$$
$$P\left(\tfrac{3}{4}T\right)|\psi\rangle = m\omega x_0|\psi\rangle, \quad X\left(T\right)|\psi\rangle = x_0|\psi\rangle.$$

4. [15] An electron is in the + state when measured in the direction $S_\theta = S_z\cos\theta + S_x\sin\theta$, so that $S_\theta|+_\theta\rangle = +\tfrac{1}{2}\hbar|+_\theta\rangle$. However, the angle θ is uncertain. In each part, it is probably a good idea to check at each step that the trace comes out correctly.

(a) [3] Suppose the angle is $\theta = \pm\tfrac{1}{3}\pi$, with equal probability for each angle. What is the state operator in the conventional $|\pm_z\rangle$ basis?

We have, from a variety of sources, the states in terms of this basis, which is

$$|+_\theta\rangle = \cos\left(\tfrac{1}{2}\theta\right)|+\rangle + \sin\left(\tfrac{1}{2}\theta\right)|-\rangle$$

The state vector is then simply taken by averaging the results for the two angles, so

$$\rho = \frac{1}{2}\sum_{\theta=\pm\frac{\pi}{3}}|+_\theta\rangle\langle+_\theta| = \frac{1}{2}\left[\begin{pmatrix}\cos\frac{\pi}{6}\\ \sin\frac{\pi}{6}\end{pmatrix}\begin{pmatrix}\cos\frac{\pi}{6} & \sin\frac{\pi}{6}\end{pmatrix} + \begin{pmatrix}\cos\frac{\pi}{6}\\ -\sin\frac{\pi}{6}\end{pmatrix}\begin{pmatrix}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6}\end{pmatrix}\right]$$
$$= \frac{1}{2}\left[\begin{pmatrix}\cos^2\frac{\pi}{6} & \cos\frac{\pi}{6}\sin\frac{\pi}{6}\\ \cos\frac{\pi}{6}\sin\frac{\pi}{6} & \sin^2\frac{\pi}{6}\end{pmatrix} + \begin{pmatrix}\cos^2\frac{\pi}{6} & -\cos\frac{\pi}{6}\sin\frac{\pi}{6}\\ -\cos\frac{\pi}{6}\sin\frac{\pi}{6} & \sin^2\frac{\pi}{6}\end{pmatrix}\right] = \begin{pmatrix}\cos^2\frac{\pi}{6} & 0\\ 0 & \sin^2\frac{\pi}{6}\end{pmatrix}$$
$$= \begin{pmatrix}\left(\frac{\sqrt{3}}{2}\right)^2 & 0\\ 0 & \left(\frac{1}{2}\right)^2\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}.$$

The result has trace one, so it's probably right.

(b) [4] Suppose the angle θ is randomly distributed in the range $-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi$, with all angles equally likely. What is the state operator in the conventional $|\pm_z\rangle$ basis?

Instead of adding two angles, we need to integrate over all angles, and divide by the range of angles, which is π, so we have

$$\rho = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}|+_\theta\rangle\langle+_\theta|d\theta = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right)\\ \sin\left(\frac{1}{2}\theta\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{1}{2}\theta\right) & \sin\left(\frac{1}{2}\theta\right)\end{pmatrix}d\theta$$
$$= \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\begin{pmatrix}\cos^2\left(\frac{1}{2}\theta\right) & \cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right)\\ \cos\left(\frac{1}{2}\theta\right)\sin\left(\frac{1}{2}\theta\right) & \sin^2\left(\frac{1}{2}\theta\right)\end{pmatrix}d\theta = \frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\begin{pmatrix}1+\cos\theta & \sin\theta\\ \sin\theta & 1-\cos\theta\end{pmatrix}d\theta$$
$$= \frac{1}{2\pi}\begin{pmatrix}\theta+\sin\theta & -\cos\theta\\ -\cos\theta & \theta-\sin\theta\end{pmatrix}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2\pi}\left[\begin{pmatrix}\frac{1}{2}\pi+1 & 0\\ 0 & \frac{1}{2}\pi-1\end{pmatrix} - \begin{pmatrix}-\frac{1}{2}\pi-1 & 0\\ 0 & -\frac{1}{2}\pi+1\end{pmatrix}\right],$$

$$\rho = \begin{pmatrix} \frac{1}{2}+\frac{1}{\pi} & 0 \\ 0 & \frac{1}{2}-\frac{1}{\pi} \end{pmatrix}.$$

Once again, the trace is one, so it's probably correct.

(c) [4] Suppose the angle θ is randomly distributed in the range $-\pi < \theta < \pi$, with all angles equally likely. What is the state operator in the conventional $|\pm_z\rangle$ basis?

This is identical to the previous part, except the range is twice as big and of course the limits of integration change, so we have

$$\rho = \frac{1}{2\pi}\int_{-\pi}^{\pi} |+_\theta\rangle\langle +_\theta| d\theta = \cdots = \frac{1}{4\pi}\int_{-\pi}^{\pi} \begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} d\theta = \frac{1}{4\pi}\begin{pmatrix} \theta+\sin\theta & -\cos\theta \\ -\cos\theta & \theta-\sin\theta \end{pmatrix}_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\begin{pmatrix} \pi & -1 \\ -1 & \pi \end{pmatrix} - \begin{pmatrix} -\pi & -1 \\ -1 & -\pi \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

So this is a completely unpolarized state operator.

(d) [4] In each of the cases listed above, what is the expectation value of S_z?

The expectation value can be found via

$$\langle S_z \rangle = \mathrm{Tr}(\rho S_z) = \tfrac{1}{2}\hbar \mathrm{Tr}(\rho\sigma_z)$$

In every case, this trace is easy to work out.

$$\langle S_z \rangle_a = \tfrac{1}{2}\hbar \mathrm{Tr}\left[\begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\left(\tfrac{3}{4}-\tfrac{1}{4}\right) = \tfrac{1}{4}\hbar,$$

$$\langle S_z \rangle_b = \tfrac{1}{2}\hbar \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2}+\frac{1}{\pi} & 0 \\ 0 & \frac{1}{2}-\frac{1}{\pi} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\left(\tfrac{1}{2}+\tfrac{1}{\pi}-\tfrac{1}{2}+\tfrac{1}{\pi}\right) = \tfrac{1}{\pi}\hbar$$

$$\langle S_z \rangle_b = \tfrac{1}{2}\hbar \mathrm{Tr}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \tfrac{1}{2}\hbar\left(\tfrac{1}{2}-\tfrac{1}{2}\right) = 0.$$

Perhaps not surprisingly, the first two cases have a positive expectation value, while the third vanishes. This is because in the first two cases, though the spin is random, it's definitely at an angle that is closer to +*z* than –*z*., but in the third it is equally likely at all angles.

5. [20] A general Hermitian operator in a two-dimensional system, such as the state vector for the spin of a spin-1/2 particle, takes the form $\rho = \frac{1}{2}(a\mathbf{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ are the Pauli matrices, 1 is the unit matrix, and r is an arbitrary three-dimensional vector.

(a) Find the eigenvalues of this matrix in general. It may be helpful to recall that $(\mathbf{r}\cdot\boldsymbol{\sigma})^2 = \mathbf{r}^2$.

It is not hard to use the hint to find the eigenvalues, but we can also do it directly. Writing $\mathbf{r} = (x, y, z)$, we see that

$$\rho = \tfrac{1}{2}\left(a\mathbf{1} + x\sigma_x + y\sigma_y + z\sigma_z\right) = \frac{1}{2}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix}$$

Ignoring the overall factor of ½, we can find the eigenvalues of the remaining matrix λ by demanding

$$0 = \det\begin{pmatrix} a+z-\lambda & x-iy \\ x+iy & a-z-\lambda \end{pmatrix} = (a-\lambda)^2 - z^2 - (x-iy)(x+iy) = (\lambda-a)^2 - x^2 - y^2 - z^2,$$

$$\lambda - a = \pm\sqrt{x^2+y^2+z^2} = \pm|\mathbf{r}|$$

Putting back in the factor of two, we have

$$\lambda = \tfrac{1}{2}\left(a \pm |\mathbf{r}|\right)$$

(b) [4] What restrictions can be placed on *a* and r if this represents a state operator?

State operators have two restrictions on their eigenvalues: they must have eigenvalues that add to one, and they must be positive. In other words, we must have

$$1 = \tfrac{1}{2}\left(a - |\mathbf{r}|\right) + \tfrac{1}{2}\left(a + |\mathbf{r}|\right) = a,$$
$$0 \le \tfrac{1}{2}\left(a \pm |\mathbf{r}|\right).$$

The first restriction implies $a = 1$. For the second, we have two constraints, but only the minus one yields any information, for which we see that $0 \le a - |\mathbf{r}|$, which implies $|\mathbf{r}| \le 1$.

$$a = 1 \quad \text{and} \quad |\mathbf{r}| \le 1$$

(c) [4] Show that all four components of *a* and r are determined if we know every component of the expectation value of the spin $\langle \mathrm{S} \rangle$.

We already automatically know that $a = 1$. As for the spin expectation values,

$$
\begin{aligned}
\langle S_i \rangle &= \tfrac{1}{2}\hbar \operatorname{Tr}(\rho\sigma_i) = \tfrac{1}{4}\hbar \operatorname{Tr}\left[(1+\mathbf{r}\cdot\boldsymbol{\sigma})\sigma_i\right] = \tfrac{1}{4}\hbar \operatorname{Tr}\left(\sigma_i + \sum_j r_j \sigma_j \sigma_i\right) \\
&= \tfrac{1}{4}\hbar \operatorname{Tr}\left[\sigma_i + \sum_j r_j \sigma_j \sigma_i\right] = \tfrac{1}{4}\hbar\left[\operatorname{Tr}(\sigma_i) + \sum_j r_j \operatorname{Tr}\left(\mathbf{1}\delta_{ij} + \sum_k i\varepsilon_{jik}\sigma_k\right)\right] \\
&= \tfrac{1}{4}\hbar\left[0 + \sum_j r_j\left(\operatorname{Tr}(\mathbf{1})\delta_{ij} + \sum_k i\varepsilon_{jik}\operatorname{Tr}(\sigma_k)\right)\right] = \tfrac{1}{4}\hbar\left[\sum_j r_j \delta_{ij} 2 + 0\right] = \tfrac{1}{2}\hbar r_i \,.
\end{aligned}
$$

This can easily be summarized as $\langle \mathbf{S} \rangle = \tfrac{1}{2}\hbar\mathbf{r}$, so we can get all three components of **r** from $\langle \mathbf{S} \rangle$.

(d) [3] Under what constraints will this density matrix be a pure state?

A pure state has eigenvalues 0 and 1 only, so we must have

$$\lambda = \tfrac{1}{2}\left(a \pm |\mathbf{r}|\right) = \tfrac{1}{2} \pm \tfrac{1}{2}|\mathbf{r}| = 0 \text{ or } 1$$

Obviously, this will happen if $|\mathbf{r}| = 1$.

(e) [5] A particle with this density matrix is under the influence of a Hamiltonian

$$H = \tfrac{1}{2}\hbar\omega\sigma_3 = \hbar\omega\begin{pmatrix} \tfrac{1}{2} & 0 \\ 0 & -\tfrac{1}{2} \end{pmatrix}$$

Find a formula for $d\mathbf{r}/dt$ and da/dt, technically four equations, one of which will be trivial.

The state operator (or density matrix) evolves according to

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}[H, \rho],$$

$$\frac{1}{2}\frac{d}{dt}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix} = \frac{1}{i\hbar}\frac{\hbar\omega}{2}\frac{1}{2}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix} - \begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\},$$

$$\frac{d}{dt}\begin{pmatrix} a+z & x-iy \\ x+iy & a-z \end{pmatrix} = \frac{\omega}{2i}\left\{\begin{pmatrix} a+z & x-iy \\ -x-iy & -a+z \end{pmatrix} - \begin{pmatrix} a+z & -x+iy \\ x+iy & -a+z \end{pmatrix}\right\} = \omega\begin{pmatrix} 0 & -ix-y \\ ix-y & 0 \end{pmatrix}.$$

Equating component by component, we get four simultaneous equations:

$$\dot{a} + \dot{z} = 0, \quad \dot{a} - \dot{z} = 0, \quad \dot{x} - i\dot{y} = -i\omega x - \omega y, \quad \dot{x} + i\dot{y} = i\omega x - \omega y \,.$$

We now need to solve these equations for each of the four time derivatives. If we add and subtract the first two, we quickly determine that *a* and *z* are unchanging over time. If we add and subtract the last two, we get a $2\dot{x} = -2\omega y$ and $2i\dot{y} = 2i\omega x$. Fortunately, these turn into real equations, and our final answer is

$$\dot{a} = \dot{z} = 0, \quad \dot{x} = -\omega y \quad \text{and} \quad \dot{y} = \omega x$$

This can be more easily summarized as $\dot{a}=0$ and $\dot{\mathbf{r}}=-\omega\hat{\mathbf{z}}\times\mathbf{r}$ if we prefer.

6. **[20] There is another version of the disproof of the "local hidden variables" hypothesis that does *not* require a discussion of probabilities. Consider a system consisting of *three* spins in the state**

$$|\psi\rangle=\tfrac{1}{\sqrt{2}}\left(|+++\rangle-|---\rangle\right)$$

Each of these spins will be measured on either the *x*-axis or the *y*-axis, that is, we will be measuring one of each of the following pairs: $\{(\sigma_{x1},\sigma_{y1}),(\sigma_{x2},\sigma_{y2}),(\sigma_{x3},\sigma_{y3})\}$. The measurements will yield three eigenvalues, which will be one each from the pairs (x_1,y_1), (x_2,y_2), (x_3,y_3). Each of these eigenvalues can take only the values ±1.

(a) [6] Consider each of the operators

$$B_1=\sigma_{x1}\sigma_{y2}\sigma_{y3},\quad B_2=\sigma_{y1}\sigma_{x2}\sigma_{y3},\quad B_3=\sigma_{y1}\sigma_{y2}\sigma_{x3}.$$

Show that $|\psi\rangle$ is an eigenstate of each of these operators, and calculate the eigenvalue.

We recall that $\sigma_x|\pm\rangle=|\mp\rangle$ and $\sigma_y|\pm\rangle=\pm i|\mp\rangle$, so

$$B_1|\psi\rangle=\tfrac{1}{\sqrt{2}}\sigma_{x1}\sigma_{y2}\sigma_{y3}\left(|+++\rangle-|---\rangle\right)=\tfrac{1}{\sqrt{2}}\left(\sigma_{x1}\sigma_{y2}\sigma_{y3}|+++\rangle-\sigma_{x1}\sigma_{y2}\sigma_{y3}|---\rangle\right)$$
$$=\tfrac{1}{\sqrt{2}}\left(i^2|---\rangle-(-i)^2|+++\rangle\right)=\tfrac{1}{\sqrt{2}}\left(|+++\rangle-|---\rangle\right)=|\psi\rangle,$$
$$B_2|\psi\rangle=\tfrac{1}{\sqrt{2}}\sigma_{y1}\sigma_{x2}\sigma_{y3}\left(|+++\rangle-|---\rangle\right)=\tfrac{1}{\sqrt{2}}\left(\sigma_{y1}\sigma_{x2}\sigma_{y3}|+++\rangle-\sigma_{y1}\sigma_{x2}\sigma_{y3}|---\rangle\right)$$
$$=\tfrac{1}{\sqrt{2}}\left(i^2|---\rangle-(-i)^2|+++\rangle\right)=\tfrac{1}{\sqrt{2}}\left(|+++\rangle-|---\rangle\right)=|\psi\rangle,$$
$$B_1|\psi\rangle=\tfrac{1}{\sqrt{2}}\sigma_{y1}\sigma_{y2}\sigma_{x3}\left(|+++\rangle-|---\rangle\right)=\tfrac{1}{\sqrt{2}}\left(\sigma_{y1}\sigma_{y2}\sigma_{x3}|+++\rangle-\sigma_{y1}\sigma_{y2}\sigma_{x3}|---\rangle\right)$$
$$=\tfrac{1}{\sqrt{2}}\left(i^2|---\rangle-(-i)^2|+++\rangle\right)=\tfrac{1}{\sqrt{2}}\left(|+++\rangle-|---\rangle\right)=|\psi\rangle.$$

In other words, it is an eigenstate of all three operators with eigenvalue +1 in each case.

(b) [3] According to quantum mechanics, suppose you happened to measure one of the three combinations B_1, B_2, or B_3. What is the prediction for the *product* of the results of those measurements, $x_1y_2y_3$, $y_1x_2y_3$, or $y_1y_2x_3$?

Well, if you measure B_1, for example, you would need to measure σ_{x1}, σ_{y2}, and σ_{y3}, and you would get the three eigenvalues x_1, y_2 and y_3. But we know the product of these operators has eigenvalue +1 when you multiply them. A similar argument goes for the other two. So in summary,

$$x_1y_2y_3=y_1x_2y_3=y_1y_2x_3=1$$

Of course, it is probably wrong to write all these equations together, since you can't measure every one of these for a single particle, according to quantum mechanics.

(c) [3] According to the philosophy of hidden variables, the values for the products you found in part (b) must be true, even if you don't make those exact measurements. Based on the three formulas you found in part (b), make a prediction for the product $x_1x_2x_3$ (hint: consider the product of the three possibilities in part (b)). ***This has nothing to do with quantum mechanics!***

The argument is simple. Each of the values that you measure on the three particles clearly will equal ±1. We have three products that equal one. If we multiply these three things together, we must still get 1. Hence

$$1=\left(x_1y_2y_3\right)\left(y_1x_2y_3\right)\left(y_1y_2x_3\right)=x_1x_2x_3y_1^2y_2^2y_3^2$$

Now, since each variable is ±1, it is clear that $y_1^2=y_2^2=y_3^2=1$. Therefore $1=x_1x_2x_3$.

This conclusion doesn't really mean anything in quantum mechanics. You can't talk about all these other variables unless you actually measure them, so no such a-priori conclusion can be reached.

(d) [3] According to the philosophy of hidden variables, the product $x_1x_2x_3$ must be what you would get if you performed all three measurements in the x-direction, and then multiplied them. Hence, the operator

$$A=\sigma_{x1}\sigma_{x2}\sigma_{x3}$$

will yield what value if you were to measure it, according to the philosophy of hidden variables?

So if we set up the experiment to measure these three operators instead, the philosophy of hidden variables demands that the product of the three measured quantities x_1, x_2, and x_3 must come out to $x_1x_2x_3=1$.

(e) [5] Show that $\left|\psi\right\rangle$ is, in fact, an eigenstate of A, but it has the wrong eigenvalue according to the prediction of part (d).

Proving it is an eigenstate is easy:

$$\begin{aligned}A\left|\psi\right\rangle&=\tfrac{1}{\sqrt{2}}\sigma_{x1}\sigma_{x2}\sigma_{x3}\left(\left|+++\right\rangle-\left|---\right\rangle\right)=\tfrac{1}{\sqrt{2}}\left(\sigma_{x1}\sigma_{x2}\sigma_{x3}\left|+++\right\rangle-\sigma_{x1}\sigma_{x2}\sigma_{x3}\left|---\right\rangle\right)\\&=\tfrac{1}{\sqrt{2}}\left(\left|---\right\rangle-\left|+++\right\rangle\right)=-\left|\psi\right\rangle.\end{aligned}$$

Hence quantum mechanics predicts that the result will be -1 if you perform this set of three measurements. Since this experiment has not been done, we don't know which of the predictions is correct, but everyone assumes quantum mechanics is right.

7. [20] In lecture I showed how MWI can account for the results when you measure a series of electrons with spin states $|+_x\rangle$. In this problem, we will instead measure the spin of electrons with spin state $|\theta\rangle$, where $|\theta\rangle = \cos\left(\tfrac{1}{2}\theta\right)|+\rangle + \sin\left(\tfrac{1}{2}\theta\right)|-\rangle$.

(a) [4] According to the Copenhagen interpretation, if we measure the spin S_z of this particle, what results might we get, and what are the probabilities of each of these results? What is the average value we would get for this result?

The eigenvalues of S_z are $\pm\tfrac{1}{2}\hbar$, and the corresponding probabilities are simply

$$P\left(+\tfrac{1}{2}\hbar\right) = \left|\langle +|\theta\rangle\right| = \cos^2\left(\tfrac{1}{2}\theta\right) \quad \text{and} \quad P\left(-\tfrac{1}{2}\hbar\right) = \left|\langle -|\theta\rangle\right| = \sin^2\left(\tfrac{1}{2}\theta\right)$$

Therefore, the average value would be

$$\langle S_z\rangle = \cos^2\left(\tfrac{1}{2}\theta\right)\left(\tfrac{1}{2}\hbar\right) + \sin^2\left(\tfrac{1}{2}\theta\right)\left(-\tfrac{1}{2}\hbar\right) = \tfrac{1}{2}\hbar\left[\cos^2\left(\tfrac{1}{2}\theta\right) - \sin^2\left(\tfrac{1}{2}\theta\right)\right] = \tfrac{1}{2}\hbar\cos\theta$$

(b) [5] Now we start working on the Multiple Worlds calculation. Define the state of N particles as $|\Psi_N\rangle = |\theta\rangle\otimes|\theta\rangle\otimes\cdots\otimes|\theta\rangle$,.Define the average spin operator $\bar{S}_z = \sum_{i=1}^N S_{Zi}\big/N$. Write out explicitly $\bar{S}_z|\Psi_N\rangle$. You may find it useful to use the state $|-\theta\rangle = \cos\left(\tfrac{1}{2}\theta\right)|+\rangle - \sin\left(\tfrac{1}{2}\theta\right)|-\rangle$ in your answer.

It's helpful to first work out

$$S_z|\theta\rangle = S_z\left[\cos\left(\tfrac{1}{2}\theta\right)|+\rangle + \sin\left(\tfrac{1}{2}\theta\right)|-\rangle\right] = \tfrac{1}{2}\hbar\left[\cos\left(\tfrac{1}{2}\theta\right)|+\rangle - \sin\left(\tfrac{1}{2}\theta\right)|-\rangle\right] = \tfrac{1}{2}\hbar|-\theta\rangle$$

We therefore find

$$\bar{S}_z|\Psi_N\rangle = \frac{1}{N}\sum_{i=1}^N S_{Zi}|\theta,\theta\cdots\theta\rangle = \frac{\hbar}{2N}\left(|-\theta,\theta\cdots\theta\rangle + |\theta,-\theta\cdots\theta\rangle + \cdots + |\theta,\theta\cdots-\theta\rangle\right).$$

(c) [7] Work out the expectation values $\langle\psi|\bar{S}_z|\psi\rangle$ and $\langle\psi|\bar{S}_z^2|\psi\rangle$. I strongly recommend you first write out expressions for the four expressions $\langle\pm\theta|\pm\theta\rangle$.

We'll take the hint, and find

$$\langle\pm\theta|\pm\theta\rangle = \left[\cos\tfrac{\theta}{2}\langle +| \pm \sin\tfrac{\theta}{2}\langle -|\right]\left[\cos\tfrac{\theta}{2}|+\rangle \pm \sin\tfrac{\theta}{2}|-\rangle\right] = \cos^2\tfrac{\theta}{2} + \sin^2\tfrac{\theta}{2} = 1,$$

$$\langle\pm\theta|\mp\theta\rangle = \left[\cos\tfrac{\theta}{2}\langle +| \pm \sin\tfrac{\theta}{2}\langle -|\right]\left[\cos\tfrac{\theta}{2}|+\rangle \mp \sin\tfrac{\theta}{2}|-\rangle\right] = \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = \cos\theta.$$

We therefore have

$$\langle\psi|\bar{S}_z|\psi\rangle = \langle\theta,\theta\cdots\theta|\frac{\hbar}{2N}\left(|-\theta,\theta\cdots\theta\rangle + |\theta,-\theta\cdots\theta\rangle + \cdots + |\theta,\theta\cdots-\theta\rangle\right)$$

$$= \frac{\hbar}{2N}\left(\cos\theta\cdot 1\cdots 1 + 1\cdot\cos\theta\cdots 1 + \cdots + 1\cdot 1\cdots\cos\theta\right) = \frac{\hbar N}{2N}\cos\theta = \frac{\hbar\cos\theta}{2}.$$

We also need

$$\langle\psi|\bar{S}_z^2|\psi\rangle = \left(\langle-\theta,\theta\cdots\theta| + \langle\theta,-\theta\cdots\theta| + \cdots + \langle\theta,\theta\cdots-\theta|\right)\left(\frac{\hbar}{2N}\right)^2$$

$$\left(|-\theta,\theta\cdots\theta\rangle + |\theta,-\theta\cdots\theta\rangle + \cdots + |\theta,\theta\cdots-\theta\rangle\right).$$

If you multiply this out, there will be N^2 terms in total. For N of them, the "odd" one will be in the same place, for which we have, for example

$$\langle-\theta,\theta\cdots\theta|-\theta,\theta\cdots\theta\rangle = 1\cdot 1\cdots 1 = 1.$$

For the remaining $N^2 - N$ of them, we have

$$\langle-\theta,\theta\cdots\theta|\theta,-\theta\cdots\theta\rangle = \cos\theta\cdot\cos\theta\cdots 1 = \cos^2\theta.$$

Adding this all together, we have

$$\langle\psi|\bar{S}_z^2|\psi\rangle = \left(\frac{\hbar}{2N}\right)^2\left[N\cdot 1 + \left(N^2 - N\right)\cos^2\theta\right] = \frac{\hbar^2}{4}\left[\cos^2\theta + \frac{1}{N}\left(1 - \cos^2\theta\right)\right]$$

$$= \frac{\hbar^2}{4}\left(\cos^2\theta + \frac{\sin^2\theta}{N}\right)$$

(d) [4] Show that $\left(\Delta\bar{S}_z\right)^2 = \hbar^2\sin^2\theta/4N$ and show that therefore, in the limit $N \to \infty$, the wave function becomes an eigenstate of $\bar{S}_z$ with the eigenvalue found in part (a).

The uncertainty in $\bar{S}_z$ is

$$\left(\Delta\bar{S}_z\right)^2 = \langle\psi|\bar{S}_z^2|\psi\rangle - \langle\psi|\bar{S}_z|\psi\rangle^2 = \frac{\hbar^2}{4}\left(\cos^2\theta + \frac{\sin^2\theta}{N}\right) - \left[\frac{\hbar\cos\theta}{2}\right]^2 = \frac{\hbar^2\sin^2\theta}{4N}.$$

In the limit $N \to \infty$, this vanishes, so we are in an eigenstate, and note that the expectation value we found in part (c) matches the value in part (a).