

## Solutions to Chapter 1

1. [10] A particle of mass  $m$  lies in one-dimension in a potential of the form  $V(x) = Fx$ , where  $F$  is constant. The wave function at time  $t$  is given by

$$\Psi(x, t) = N(t) \exp\left[-\frac{1}{2} A(t) x^2 + B(t) x\right]$$

where  $N$ ,  $A$ , and  $B$  are all complex functions of time. Use Schrödinger's equation to derive equations for the time derivative of the three functions  $A$ ,  $B$ , and  $N$ . You do not need to solve these equations.

We first work out the time derivative and two space derivatives.

$$\begin{aligned} \frac{\partial}{\partial t} \psi(x) &= \left(\dot{N} - \frac{1}{2} N \dot{A} x^2 + N \dot{B} x\right) \exp\left(-\frac{1}{2} A x^2 + B x\right), \\ \frac{\partial}{\partial x} \psi(x) &= N(-A x + B) \exp\left(-\frac{1}{2} A x^2 + B x\right), \\ \frac{\partial^2}{\partial x^2} \psi(x) &= N \left[-A + (-A x + B)^2\right] \exp\left(-\frac{1}{2} A x^2 + B x\right), \end{aligned}$$

Now we simply plug these results into Schrödinger's equation:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi &= -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(x) \psi, \\ i\hbar \left(\dot{N} - \frac{1}{2} N \dot{A} x^2 + N \dot{B} x\right) \exp\left(-\frac{1}{2} A x^2 + B x\right) &= -\frac{\hbar^2}{2m} N \left[-A + (-A x + B)^2\right] \exp\left(-\frac{1}{2} A x^2 + B x\right) \\ &\quad + FxN \exp\left(-\frac{1}{2} A x^2 + B x\right). \end{aligned}$$

Canceling the common exponential and dividing by  $i\hbar N$ , this simplifies to

$$\frac{\dot{N}}{N} - \frac{1}{2} \dot{A} x^2 + \dot{B} x = \frac{i\hbar}{2m} (A^2 x^2 - 2ABx + B^2 - A) - \frac{iF}{\hbar} x.$$

This expression must be true at all positions  $x$ . The only way this can happen is if the coefficients of  $x^2$ ,  $x$ , and the constant terms all match on the two sides of the equation. This implies

$$\dot{A} = -\frac{i\hbar}{m} A^2, \quad \dot{B} = -\frac{i\hbar}{m} AB - \frac{iF}{\hbar}, \quad \dot{N} = iN \frac{\hbar}{2m} (B^2 - A).$$

The first of these is, in fact, pretty easy to solve, but the others are a bit trickier.

$$\frac{1}{A(t)} = \frac{1}{A_0} + \frac{i\hbar}{m} t.$$

2. [10] For each of the wave functions in one dimension given below,  $N$  and  $a$  are positive real numbers. Determine the normalization constant  $N$  in terms of  $a$ , and determine the probability that a measurement of the position of the particle will yield  $x > a$ .

(a) [4]  $\psi(x) = N/(x+ia)$

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = \int_{-\infty}^{\infty} \frac{N^2 dx}{(x+ia)(x-ia)} = \int_{-\infty}^{\infty} \frac{N^2 dx}{x^2 + a^2} = \frac{N^2}{a} \tan^{-1}\left(\frac{x}{a}\right) \Big|_{-\infty}^{\infty} = \frac{\pi N^2}{a},$$

$$N = \sqrt{a/\pi}$$

$$P(x > a) = \left(N^2/a\right) \tan^{-1}(x/a) \Big|_a^{\infty} = \frac{\frac{1}{2}\pi - \frac{1}{4}\pi}{\pi} = \frac{1}{4} = 25\%$$

(b) [3]  $\psi(x) = N \exp(-|x|/a)$

$$1 = N^2 \int_{-\infty}^{\infty} e^{-2|x|/a} dx = 2N^2 \int_0^{\infty} e^{-2x/a} dx = N^2 a e^{-2x/a} \Big|_0^{\infty} = N^2 a,$$

$$N = 1/\sqrt{a},$$

$$P(x > a) = N^2 \int_a^{\infty} e^{-2|x|/a} dx = \frac{1}{2} N^2 a e^{-2x/a} \Big|_a^{\infty} = \frac{1}{2} e^{-2} = 6.767\% .$$

(c) [3]  $\psi(x) = \begin{cases} Nx^2(x-2a) & \text{for } 0 < x < 2a, \\ 0 & \text{otherwise.} \end{cases}$

$$1 = N^2 \int_0^{2a} [x^2(x-2a)]^2 dx = N^2 \int_0^{2a} (x^6 - 4ax^5 + 4a^2x^4) dx$$

$$= N^2 \left( \frac{1}{7}x^7 - \frac{2}{3}ax^6 + \frac{4}{5}a^2x^5 \right) \Big|_0^{2a} = \frac{128}{105} N^2 a^7 ,$$

$$N = \sqrt{\frac{105}{128}} a^{-7/2} .$$

$$P(x > a) = N^2 \int_0^{2a} [x^2(x-2a)]^2 dx = N^2 \left( \frac{1}{7}x^7 - \frac{2}{3}ax^6 + \frac{4}{5}a^2x^5 \right) \Big|_a^{2a} = \frac{105}{128} a^{-7} \left( \frac{128}{105} a^7 - \frac{29}{105} a^7 \right)$$

$$= \frac{99}{128} = 77.34\%$$

3. An electron in the ground state of hydrogen has, in spherical coordinates, the wave function  $\psi(r, \theta, \phi) = Ne^{-r/a}$  where  $N$  and  $a$  are positive constants.

Determine the normalization constant  $N$  and the probability that a measurement of the position will yield  $r > a$ . Don't forget you are working in three dimensions!

In three dimensions, when working in spherical coordinates, the normalization condition is

$$1 = \iiint |\psi(\mathbf{r})|^2 d^3\mathbf{r} = \int_0^\infty r^2 dr \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\phi |\psi(\mathbf{r})|^2$$

In this case, the wave function depends only on  $r$ , so the inner two integrals are trivial.

$$1 = 4\pi N^2 \int_0^\infty r^2 e^{-2r/a} dr = 4\pi N^2 \left( -\frac{1}{2} r^2 a - \frac{1}{2} r a^2 - \frac{1}{4} a^3 \right) e^{-2r/a} \Big|_0^\infty = \pi a^3 N^2$$

$$N = 1/\sqrt{\pi a^3}$$

$$P(r > a) = 4\pi N^2 \int_a^\infty r^2 e^{-2r/a} dr = \frac{4}{a^3} \left( -\frac{1}{2} r^2 a - \frac{1}{2} r a^2 - \frac{1}{4} a^3 \right) e^{-2r/a} \Big|_a^\infty = 5e^{-2} = 67.67\%$$

4. [10] For each of the normalized wave functions given below, find the Fourier transform  $\tilde{\psi}(k)$ , and check that it satisfies the normalization condition

$$\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = 1.$$

(a) [5]  $\psi(x) = (A/\pi)^{1/4} \exp(iKx - \frac{1}{2}Ax^2)$

This turns into a Gaussian of the type you can find in Appendix A:

$$\begin{aligned} \tilde{\psi}(k) &= \left(\frac{A}{\pi}\right)^{1/4} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp(iKx - \frac{1}{2}Ax^2) \exp(-ikx) \\ &= \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp\left[i(K-k)x - \frac{1}{2}Ax^2\right] = \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{A/2}} \exp\left\{\frac{[i(K-k)]^2}{4(A/2)}\right\} \\ &= (\pi A)^{-1/4} \exp\left[-(K-k)^2/2A\right] \end{aligned}$$

We check this by simply seeing if the normalization works out:

$$\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} \exp\left[-(K-k)^2/A\right] dk = \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} e^{-k^2/A} dk = \frac{1}{\sqrt{\pi A}} \sqrt{\frac{\pi}{1/A}} = 1.$$

(b) [5]  $\psi(x) = \sqrt{\alpha} \exp(-\alpha|x|)$

I find absolute value problems easiest to deal with by dividing the integral into two pieces, then letting  $x \rightarrow -x$  on half of it. It is helpful to know  $\int_0^{\infty} e^{-ax} dx = 1/a$  if  $a$  has a real positive part.

$$\begin{aligned} \tilde{\psi}(k) &= \sqrt{\alpha} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\alpha|x|} e^{-ikx} = \sqrt{\frac{\alpha}{2\pi}} \left( \int_0^{\infty} e^{-\alpha x - ikx} dx + \int_{-\infty}^0 e^{\alpha x - ikx} dx \right) = \sqrt{\frac{\alpha}{2\pi}} \left( \int_0^{\infty} e^{-\alpha x - ikx} dx + \int_0^{\infty} e^{-\alpha x + ikx} dx \right) \\ &= \sqrt{\frac{\alpha}{2\pi}} \left( \frac{1}{\alpha + ik} + \frac{1}{\alpha - ik} \right) = \sqrt{\frac{\alpha}{2\pi}} \frac{2\alpha}{\alpha^2 + k^2}. \end{aligned}$$

Once again, we check it, using the trig substitution  $k = \alpha \tan \theta$  to complete the integral.

$$\int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = \frac{4\alpha^3}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{(k^2 + \alpha^2)^2} = \frac{2\alpha^3}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\alpha \sec^2 \theta d\theta}{(\alpha^2 \tan^2 \theta + \alpha^2)^2} = \frac{2}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2 \theta d\theta = 1.$$

5. [10] For each of the wave functions in question 4, find  $\bar{x}$ ,  $\Delta x$ ,  $\bar{p}$ ,  $\Delta p$ , and check that the uncertainty relationship  $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$  is satisfied.

We simply work out each case in a straightforward manner. For part (a), we have

$$\begin{aligned}\bar{x} &= \int_{-\infty}^{\infty} x \psi^*(x) \psi(x) dx = \sqrt{\frac{A}{\pi}} \int_{-\infty}^{\infty} x \exp(-iKx - \frac{1}{2}Ax^2) \exp(iKx - \frac{1}{2}Ax^2) dx \\ &= \sqrt{\frac{A}{\pi}} \int_{-\infty}^{\infty} x e^{-Ax^2} dx = 0, \\ (\Delta x)^2 &= \int_{-\infty}^{\infty} (x - \bar{x})^2 \psi^*(x) \psi(x) dx = \sqrt{\frac{A}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-Ax^2} dx = 0 = \sqrt{\frac{A}{\pi}} \Gamma\left(\frac{3}{2}\right) A^{-3/2} = \sqrt{\frac{A}{\pi}} \frac{1}{2} \sqrt{\pi} A^{-3/2} \\ &= \frac{1}{2A}, \\ \bar{p} &= \frac{\hbar}{\sqrt{\pi A}} \int_{-\infty}^{\infty} k \tilde{\psi}^*(k) \psi(k) dk = \frac{\hbar}{\sqrt{\pi A}} \int_{-\infty}^{\infty} k \exp\left[-\frac{(k-K)^2}{A}\right] dk \\ &= \frac{\hbar}{\sqrt{\pi A}} \int_{-\infty}^{\infty} (k+K) \exp(-k^2/A) dk = \frac{\hbar}{\sqrt{\pi A}} [0 + K\sqrt{\pi A}] = \hbar K, \\ (\Delta p)^2 &= \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} (\hbar k - \hbar K)^2 \tilde{\psi}^*(k) \psi(k) dk = \frac{\hbar^2}{\sqrt{\pi A}} \int_{-\infty}^{\infty} (k-K)^2 \exp\left[-(k-K)^2/A\right] dk \\ &= \frac{\hbar^2}{\sqrt{\pi A}} \int_{-\infty}^{\infty} k^2 \exp(-k^2/A) dk = \frac{\hbar^2}{\sqrt{\pi A}} \Gamma\left(\frac{3}{2}\right) A^{3/2} = \frac{\hbar^2}{\sqrt{\pi A}} \frac{1}{2} \sqrt{\pi} A^{3/2} = \frac{1}{2} \hbar^2 A\end{aligned}$$

In summary,  $\Delta x = 1/\sqrt{2A}$ ,  $\Delta p = \hbar\sqrt{A/2}$ , and  $(\Delta x)(\Delta p) = \frac{1}{2}\hbar$ , so the inequality is just barely satisfied. For part (b), we have

$$\begin{aligned}\bar{x} &= \alpha \int_{-\infty}^{\infty} x e^{-2\alpha|x|} dx = \alpha \left( \int_0^{\infty} x e^{-2\alpha x} dx + \int_{-\infty}^0 x e^{2\alpha x} dx \right) = \alpha \left( \int_0^{\infty} x e^{-2\alpha x} dx - \int_0^{\infty} x e^{-2\alpha x} dx \right) = 0 \\ (\Delta x)^2 &= \alpha \int_{-\infty}^{\infty} x^2 e^{-2\alpha|x|} dx = \alpha \left( \int_0^{\infty} x^2 e^{-2\alpha x} dx + \int_{-\infty}^0 x^2 e^{2\alpha x} dx \right) = 2\alpha \int_0^{\infty} x^2 e^{-2\alpha x} dx \\ &= \frac{2\alpha}{(2\alpha)^3} \int_0^{\infty} w^2 e^{-w} dw = \frac{2}{4\alpha^2} = \frac{1}{2\alpha^2} \\ \bar{p} &= \frac{\hbar\alpha(2\alpha)^2}{2\pi} \int_{-\infty}^{\infty} \frac{k dk}{(k^2 + \alpha^2)^2} = \frac{2\hbar\alpha^3}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha \tan \theta \alpha \sec^2 \theta d\theta}{(\alpha^2 \tan^2 \theta + \alpha^2)^2} = \frac{2\hbar\alpha}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta = 0 \\ (\Delta p)^2 &= \frac{\alpha(2\alpha)^2}{2\pi} \int_{-\infty}^{\infty} \frac{(\hbar k)^2 dk}{(k^2 + \alpha^2)^2} = \frac{2\hbar^2\alpha^3}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha^2 \tan^2 \theta \alpha \sec^2 \theta d\theta}{(\alpha^2 \tan^2 \theta + \alpha^2)^2} = \frac{2\hbar^2\alpha^2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \theta d\theta \\ &= \frac{2\hbar^2\alpha^2}{\pi} \frac{\pi}{2} = \hbar^2\alpha^2\end{aligned}$$

In summary,  $\Delta x = 1/\alpha\sqrt{2}$ ,  $\Delta p = \hbar\alpha$ , and  $(\Delta x)(\Delta p) = \hbar/\sqrt{2}$ , which also works.

**6. [10] A particle of mass  $m$  lies in the harmonic oscillator potential, given by  $V(x) = \frac{1}{2}m\omega^2 x^2$ . Later we will solve this problem exactly, but for now, we only want an approximate solution.**

**(a) [4] Let the uncertainty in the position be  $\Delta x = a$ . What is the corresponding minimum uncertainty in the momentum  $\Delta p$ ? Write an expression for the total energy (kinetic plus potential) as a function of  $a$ .**

By the uncertainty principle,  $(\Delta x)(\Delta p) \geq \frac{1}{2}\hbar$ , so if  $\Delta x = a$ , then  $\Delta p \geq \hbar/2a$ . The formula for the energy is

$$E = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

Now, the minimum energy classically would occur when  $p = 0$  and  $x = 0$ , but this is impossible quantum mechanically because we cannot know them exactly. Assuming  $x$  and  $p$  actually take on values approximately equal to their uncertainties, the corresponding energy would be

$$E \approx \frac{\hbar^2}{8ma^2} + \frac{1}{2}m\omega^2 a^2$$

**(b) [6] Find the minimum of the energy function you found in (a), and thereby estimate the minimum energy (called zero point energy) for a particle in a harmonic oscillator. Your answer should be very simple.**

To find the minimum energy, we simply take the derivative of the function we just found and set it to zero

$$\begin{aligned} 0 &= \frac{dE}{da} = -\frac{\hbar^2}{4ma^3} + m\omega^2 a, \\ \hbar^2 &= 4m^2 \omega^3 a^4 \\ a^2 &= \frac{\hbar}{2m\omega} \end{aligned}$$

You now simply plug this energy back in to find

$$E = \frac{\hbar^2}{8m} \frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega$$

This answer is, in fact, exactly correct, and the derivation can be shown to be exact as well, but this is a coincidence special to the harmonic oscillator.