Physics 741 – Graduate Quantum Mechanics 1

Solution Set W

1. [25] Suppose an electron lies in a region with electric and magnetic fields:

$$\mathbf{B} = B\hat{\mathbf{k}}$$

$$\mathbf{E} = \frac{m\omega_0^2}{e} x\hat{\mathbf{i}}$$

(a) [2] Find the electric potential $U(x)$ such that $\mathbf{E} = -\nabla U(x)$ that could lead to this electric field.

We need the potential to get the derivative in the *x*-direction to yield $-m\omega_0^2 x/e$, which tells us that the correct choice is

$$U(x) = -\frac{m\omega_0^2}{2e} x^2$$

This is easily checked.

(b) [3] The magnetic field is independent of translations in all three dimensions. However, the electrostatic potential is independent of translations in only two of those dimensions. Find a vector potential A with $\mathbf{B} = \nabla \times \mathbf{A}$ which has translation symmetry in the *same* two directions.

There are always multiple ways to choose to write the vector potential. The electric potential is translation invariant in the *y*- and *z*-directions, so it makes a lot of sense to try to make our vector potential independent of these two coordinates as well. This means when we write $\mathbf{B} = \nabla \times \mathbf{A}$, we're going to need to get the magnetic field from taking derivatives in the *x*-direction. The way the curl works, this will work out if we choose the magnetic field to lie in the *y*-direction, and it isn't hard to see that this works if

$$\mathbf{A} = Bx\hat{\mathbf{j}}$$

(c) [4] Write out the Hamiltonian for this system. Eliminate *B* in terms of the cyclotron frequency $\omega_B = eB/m$. What two translation operators commute with this Hamiltonian? What spin operator commutes with this Hamiltonian?

The Hamiltonian is

$$H = \frac{1}{2m}\left(\mathbf{P}+e\mathbf{A}\right)^2 - eU + \frac{ge}{2m}\mathbf{B}\cdot\mathbf{S} = \frac{1}{2m}\left[P_x^2 + \left(P_y + eBQ_x\right)^2 + P_z^2\right] + \tfrac{1}{2}m\omega_0^2Q_x^2 + \frac{ge}{2m}BS_z$$
$$= \frac{1}{2m}\left[P_x^2 + \left(P_y + m\omega_B Q_x\right)^2 + P_z^2\right] + \tfrac{1}{2}m\omega_0^2Q_x^2 + \tfrac{1}{2}g\omega_B S_z$$

This commutes with P_y, P_z, and S_z. Life is good.

(d) [3] Write your wave function in the form

$$\psi(\mathbf{r}) = X(x)Y(y)Z(z)\left|m_s\right\rangle$$

Based on some of the operators you worked out in part (c), deduce the form of two of the unknown functions.

Since our wave function commutes with P_y and P_z, we can choose it to be eigenstates of two of these operators, and consequently they will look like

$$Y(y) = e^{ik_y y}$$
$$Z(z) = e^{ik_z z}$$

These will have eigenvalues $\hbar k_y$ and $\hbar k_z$ under these two operators.

(e) [3] Replace the various operators by their eigenvalues in the Hamiltonian. The non-constant terms should be identifiable as a shifted harmonic oscillator.

Replacing the operators by their eigenvalues, the Hamiltonian becomes

$$H = \frac{1}{2m}\left[P_x^2 + \left(\hbar k_y + m\omega_B Q_x\right)^2 + \hbar^2 k_z^2\right] + \tfrac{1}{2}m\omega_0^2Q_x^2 + \tfrac{1}{2}g\hbar\omega_B m_s$$
$$= \frac{P_x^2}{2m} + \tfrac{1}{2}m\left(\omega_B^2 + \omega_0^2\right)Q_x^2 + \hbar k_y\omega_B Q_x + \frac{\hbar^2k_y^2}{2m} + \frac{\hbar^2k_z^2}{2m} + \tfrac{1}{2}g\hbar\omega_B m_s$$

The last few terms are constants, and the rest is simply a shifted harmonic oscillator.

(f) [4] Make a simple coordinate replacement that shifts it back. If your formulas match mine up to now, they should look like:

$$Q_x = Q_x' - \frac{\hbar k_y\omega_B}{m\left(\omega_B^2 + \omega_0^2\right)}$$

We try the substitution suggested, and find

$$H=\frac{P_x^2}{2m}+\tfrac{1}{2}m\left(\omega_B^2+\omega_0^2\right)\left[Q_x'-\frac{\hbar k_y\omega_B}{m\left(\omega_B^2+\omega_0^2\right)}\right]^2+\hbar k_y\omega_B\left[Q_x'-\frac{\hbar k_y\omega_B}{m\left(\omega_B^2+\omega_0^2\right)}\right]$$
$$+\frac{\hbar^2k_y^2}{2m}+\frac{\hbar^2k_z^2}{2m}+\tfrac{1}{2}g\hbar\omega_Bm_s$$
$$=\frac{P_x^2}{2m}+\tfrac{1}{2}m\left(\omega_B^2+\omega_0^2\right)Q_x'^2-\frac{\hbar^2k_y^2\omega_B^2}{2m\left(\omega_B^2+\omega_0^2\right)}+\frac{\hbar^2k_y^2}{2m}+\frac{\hbar^2k_z^2}{2m}+\tfrac{1}{2}g\hbar\omega_Bm_s$$

(g) [3] Find the energies of the Hamiltonian

The first two terms are simply a Harmonic oscillator, now not shifted, and the energies are just $\hbar\omega\left(n+\tfrac{1}{2}\right)$, where $\omega=\sqrt{\omega_0^2+\omega_B^2}$. Therefore the energies are in total

$$E=\hbar\sqrt{\omega_B^2+\omega_0^2}\left(n+\tfrac{1}{2}\right)+\frac{\hbar^2}{2m}\left(k_z^2+k_y^2\frac{\omega_0^2}{\omega_B^2+\omega_0^2}\right)+\tfrac{1}{2}g\hbar\omega_Bm_s$$

(h) [3] Check that they give sensible answers in the two limits when there is no electric field (pure Landau levels) or no magnetic fields (pure harmonic oscillator plus *y*- and *z*-motion)

If there are no electric fields, then $\omega_0=0$, and we have

$$E=\frac{\hbar^2k_z^2}{2m}+\hbar\omega_B\left(n+\tfrac{1}{2}+\tfrac{1}{2}gm_s\right)$$

This is exactly what we would expect. If there are no magnetic fields, then $\omega_B=0$, and we have

$$E=\hbar\omega_0\left(n+\tfrac{1}{2}\right)+\frac{\hbar^2}{2m}\left(k_z^2+k_y^2\right)$$

which is a harmonic oscillator added to motion in the *y*- and *z*-direction.