Physics 741 – Graduate Quantum Mechanics 1

Solution Set V

1. [15] This problem has nothing to do with quantum mechanics. In the presence of a charged plasma, it is possible to create electromagnetic waves that are longitudinal, having electric polarization parallel to the direction of propagation. The fields take the form

$$\mathbf{E}(\mathbf{r},t) = E_0 \hat{\mathbf{k}} \cos(kz - \omega t), \quad \mathbf{B}(\mathbf{r},t) = 0 .$$

(a) [5] Show that this electric field (and lack of magnetic field) can be written purely in terms of a vector potential, without the use of the scalar potential; that

$$\mathbf{A}_1(\mathbf{r},t) = \hat{\mathbf{k}} A_1(\mathbf{r},t) \quad \text{and} \quad U_1(\mathbf{r},t) = 0$$

and determine the function $A_1(\mathbf{r},t)$ that makes this work.

We need to find a vector field such that $\mathbf{E} = -\partial \mathbf{A}_1/\partial t$, since there is no scalar potential. Since we want **E** to come out looking like a cosine, it makes sense to try letting $\mathbf{A}_1$ look like a sine. Clearly, the time derivative isn't gong to change the direction of **A**, so it makes sense to try expressions like $\hat{\mathbf{k}} \sin(kz - \omega t)$. Fiddling a bit, we quickly find the correct solution:

$$\mathbf{A}_1(\mathbf{r},t) = E_0 \hat{\mathbf{k}} \sin(kz - \omega t)/\omega \quad \text{and} \quad U_1(\mathbf{r},t) = 0$$

We now simply check that this works

$$\mathbf{B}_1(\mathbf{r},t) = \nabla \times \mathbf{A}_1(\mathbf{r},t) = E_0 \left[\nabla \times \hat{\mathbf{k}} \sin(kz - \omega t) \right] / \omega = 0,$$

$$\mathbf{E}_1(\mathbf{r},t) = -\frac{E_0}{\omega} \frac{\partial}{\partial t} \left[\hat{\mathbf{k}} \sin(kz - \omega t) \right] - \nabla(0) = E_0 \hat{\mathbf{k}} \cos(kz - \omega t).$$

So it works.

(b) [5] Show that the same electric fields (and lack of magnetic field) can also be written purely in terms of a scalar potential,

$$\mathbf{A}_2(\mathbf{r},t) = 0 \quad \text{and} \quad U_2(\mathbf{r},t)$$

and determine the scalar potential $U_2(\mathbf{r},t)$ that makes this work.

This time, we need to get $\mathbf{E} = -\nabla U_1$. To make the gradient not vanish in the *z*-direction, we need U_1 to vary in the *z*-direction. Since we are taking a derivative, it again makes sense to try a sine function, something like $\sin(kz - \omega t)$. A bit of experimenting then tells you that the correct answer is

$$\mathbf{A}_2(\mathbf{r},t)=0 \quad \text{and} \quad U_2(\mathbf{r},t)=-E_0\sin(kz-\omega t)/k$$

That this is correct is demonstrated simply by testing it:

$$\mathbf{B}_1(\mathbf{r},t)=\nabla\times(0)=0,$$
$$\mathbf{E}_1(\mathbf{r},t)=-\frac{\partial}{\partial t}0+\frac{E_0}{k}\nabla\sin(kz-\omega t)=E_0\hat{\mathbf{k}}\cos(kz-\omega t).$$

(c) [5] Show that these two sets of potential, $(\mathbf{A}_1,U_1)$ and $(\mathbf{A}_2,U_2)$, are related by a gauge transformation, and determine explicitly the form of the gauge function $\chi(\mathbf{r},t)$ that relates them.

We need to find a single function $\chi(\mathbf{r},t)$ such that

$$\mathbf{A}_2(\mathbf{r},t)=\mathbf{A}_1(\mathbf{r},t)+\nabla\chi(\mathbf{r},t)$$
$$U_2(\mathbf{r},t)=U_1(\mathbf{r},t)-\partial\chi(\mathbf{r},t)/\partial t$$

Plugging in our specific values for each of these, we need

$$\nabla\chi(\mathbf{r},t)=-E_0\hat{\mathbf{k}}\sin(kz-\omega t)/\omega$$
$$\partial\chi(\mathbf{r},t)/\partial t=E_0\sin(kz-\omega t)/k$$

From the first expression, we see that we want something that varies in the *z*-direction, and whose derivative is a sine function, which suggests a cosine. From the second, we see that we want something whose derivative is a sine, which again suggests a cosine. A bit of trial and error will then yield

$$\chi(\mathbf{r},t)=\frac{E_0}{k\omega}\cos(kz-\omega t)$$

Simple substitution will demonstrate that this works in both equations.

$$\nabla\chi(\mathbf{r},t)=\frac{E_0}{k\omega}\hat{\mathbf{k}}\frac{\partial}{\partial z}\cos(kz-\omega t)=-\hat{\mathbf{k}}\frac{E_0}{\omega}\sin(kz-\omega t)$$
$$\frac{\partial}{\partial t}\chi(\mathbf{r},t)=\frac{E_0}{k\omega}\frac{\partial}{\partial t}\cos(kz-\omega t)=\frac{E_0}{k}\sin(kz-\omega t)$$

2. **[20] In chapter three, we defined the probability density ρ and probability current j as**

$$\rho = \Psi^* \Psi \quad \text{and} \quad \mathbf{j} = \hbar\left(-i\Psi^*\nabla\Psi + i\Psi\nabla\Psi^*\right)\big/2m = \left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right)\big/2m$$

and then derived the conservation of probability formula

$$\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$$

from Schrödinger's equation (3.1b). However, Schrödinger's equation has just changed into (10.23), and our proof is no longer valid.

(a) [3] Which of the quantities ρ and j is invariant under a gauge transformation?

Under a gauge transformation, $\Psi \to \Psi' = \Psi\exp\left[-ie\chi/\hbar\right]$, so

$$\rho' = \Psi'^*\Psi' = \Psi^*\exp\left(ie\chi/\hbar\right)\exp\left(-ie\chi/\hbar\right)\Psi = \Psi^*\Psi = \rho,$$

$$\begin{aligned}
\mathbf{j}' &= \hbar\left(-i\Psi'^*\nabla\Psi' + i\Psi'\nabla\Psi'^*\right)\big/2m \\
&= \frac{\hbar}{2m}\left\{\begin{matrix} -i\Psi^*\exp\left(ie\chi/\hbar\right)\nabla\left[\Psi\exp\left(-ie\chi/\hbar\right)\right] \\ +i\Psi\exp\left(-ie\chi/\hbar\right)\nabla\left[\Psi^*\exp\left(ie\chi/\hbar\right)\right]\end{matrix}\right\} \\
&= \frac{1}{2m}\left\{\begin{matrix} \Psi^*\exp\left(ie\chi/\hbar\right)\left[-i\hbar\left(\nabla\Psi\right)\exp\left(-ie\chi/\hbar\right) - e\Psi\exp\left(-ie\chi/\hbar\right)\left(\nabla\chi\right)\right] \\ +\Psi\exp\left(-ie\chi/\hbar\right)\left[i\hbar\left(\nabla\Psi^*\right)\exp\left(ie\chi/\hbar\right) - e\Psi^*\exp\left(ie\chi/\hbar\right)\left(\nabla\chi\right)\right]\end{matrix}\right\} \\
&= \frac{1}{2m}\left(-i\hbar\Psi^*\nabla\Psi + i\hbar\Psi\nabla\Psi^* - 2e\Psi^*\Psi\nabla\chi\right) = \mathbf{j} - \frac{e}{m}\Psi\Psi^*\nabla\chi
\end{aligned}$$

Obviously, the probability density is gauge invariant, but the current is not.

(b) [3] Show that the derivation of the conservation law is no longer valid.

There is a sign error in the original derivation, so I suppose it was never valid in the first place. We start with Schrödinger's equation.

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(\mathbf{P} + e\mathbf{A}\right)^2\Psi - eU\Psi$$

We now take the complex conjugate of this expression. The momentum **P** has a factor of *i* hidden in it, so this changes sign, but **A** stays the same.

$$-i\hbar\frac{\partial\Psi^*}{\partial t} = \frac{1}{2m}\left(\mathbf{P} - e\mathbf{A}\right)^2\Psi^* - eU\Psi^*$$

Multiply the first equation by Ψ^* and the second by Ψ, then subtract the results.

$$i\hbar\left(\Psi^*\frac{\partial\Psi}{\partial t}+\Psi\frac{\partial\Psi^*}{\partial t}\right)=\frac{1}{2m}\left\{\Psi^*\left(\mathbf{P}+e\mathbf{A}\right)^2\Psi-\Psi\left(\mathbf{P}-e\mathbf{A}\right)^2\Psi^*\right\}-eU\left(\Psi\Psi^*-\Psi^*\Psi\right),$$

$$i\hbar\frac{\partial}{\partial t}\left(\Psi^*\Psi\right)=\frac{1}{2m}\left\{\begin{array}{l}\Psi^*\left(\mathbf{P}^2-e\mathbf{A}\cdot\mathbf{P}-e\mathbf{P}\cdot\mathbf{A}+e^2\mathbf{A}^2\right)\Psi\\-\Psi\left(\mathbf{P}^2-e\mathbf{A}\cdot\mathbf{P}-e\mathbf{P}\cdot\mathbf{A}+e^2\mathbf{A}^2\right)\Psi^*\end{array}\right\}$$

$$i\hbar\frac{\partial}{\partial t}\rho=\frac{1}{2m}\left\{\begin{array}{l}\Psi^*\mathbf{P}^2\Psi+e\Psi^*\left(\mathbf{A}\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{A}\right)\Psi\\-\Psi\mathbf{P}^2\Psi^*+e\Psi\left(\mathbf{A}\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{A}\right)\Psi^*\end{array}\right\}$$

The terms with $\mathbf{A}\cdot\mathbf{P}+\mathbf{P}\cdot\mathbf{A}$ are new terms that we didn't have before. Rewriting the derivatives as gradients, this becomes

$$i\hbar\frac{\partial}{\partial t}\rho=\frac{\hbar^2}{2m}\left(\Psi\nabla^2\Psi^*-\Psi^*\nabla^2\Psi\right)-\frac{ie\hbar}{2m}\left[\begin{array}{l}\Psi^*\mathbf{A}\cdot\nabla\Psi+\Psi^*\nabla\cdot\left(\mathbf{A}\Psi\right)\\+\Psi\mathbf{A}\cdot\nabla\Psi^*+\Psi\nabla\cdot\left(\mathbf{A}\Psi^*\right)\end{array}\right],$$

$$\frac{\partial}{\partial t}\rho=-\frac{i\hbar}{2m}\left(\Psi\nabla^2\Psi^*-\Psi^*\nabla^2\Psi\right)-\frac{e}{m}\left[\Psi^*\mathbf{A}\cdot\nabla\Psi+\mathbf{A}\cdot\nabla\Psi^*+\Psi^*\Psi\left(\nabla\cdot\mathbf{A}\right)\right].$$

The first term on the right, through the usual trick, can be rewritten as

$$\begin{aligned}\nabla\cdot\left(\Psi\nabla\Psi^*-\Psi^*\nabla\Psi\right)&=\left(\nabla\Psi\right)\cdot\left(\nabla\Psi^*\right)+\Psi\nabla^2\Psi^*-\left(\nabla\Psi^*\right)\cdot\left(\nabla\Psi\right)-\Psi^*\nabla^2\Psi\\&=\Psi\nabla^2\Psi^*-\Psi^*\nabla^2\Psi\end{aligned}$$

The remaining terms are a total divergence of the expression $\Psi^*\mathbf{A}\Psi$, so we have

$$\begin{aligned}\frac{\partial}{\partial t}\rho&=-\nabla\cdot\left\{\frac{i\hbar}{2m}\left(\Psi\nabla\Psi^*-\Psi^*\nabla\Psi\right)+\frac{e}{m}\Psi^*\mathbf{A}\Psi\right\}\\&=-\nabla\cdot\left[\frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi-\Psi\mathbf{P}\Psi^*\right)+\frac{e}{m}\Psi^*\mathbf{A}\Psi\right]\end{aligned}$$

Okay, we went through a lot of work, but the good news is that when we get to part (d), we see that we are nearly done. If the last term weren't there, then we could complete this part and show that the proof *does* still work, but the extra term doesn't vanish, nor would we expect it to have no divergence.

(c) [5] Redefine the probability current j by replacing $\mathbf{P}\to\boldsymbol{\pi}$, defined by

$$\boldsymbol{\pi}\Psi=\left(\mathbf{P}+e\mathbf{A}\right)\Psi\quad\text{and}\quad\boldsymbol{\pi}\Psi^*=\left(\mathbf{P}-e\mathbf{A}\right)\Psi^*$$

Show that the new probability current j *is* gauge invariant.

The new probability current **j** is

$$\mathbf{j}=\frac{1}{2m}\left[\Psi^*\left(\mathbf{P}+e\mathbf{A}\right)\Psi-\Psi\left(\mathbf{P}-e\mathbf{A}\right)\Psi^*\right]=\frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi-\Psi\mathbf{P}\Psi^*\right)+\frac{e}{m}\Psi^*\mathbf{A}\Psi$$

Under a gauge transformation, we already have worked out in part (a) that this gets an added term of $-e\Psi\Psi^*\nabla\chi/m$. When working with the new **j**, there will also be a new term coming from the change of **A**

$$\begin{aligned}
\mathbf{j}' &= \frac{1}{2m}\left(\Psi'^*\mathbf{P}\Psi' - \Psi'\mathbf{P}\Psi'^*\right) + \frac{e}{m}\mathbf{A}'\Psi'^*\Psi' \\
&= \frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right) - \frac{e}{m}\nabla\chi\Psi^*\Psi + \frac{e}{m}\left(\mathbf{A}+\nabla\chi\right)\Psi^*\exp\left(\frac{ie\chi}{\hbar}\right)\Psi\exp\left(-\frac{ie\chi}{\hbar}\right) \\
&= \frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right) - \frac{e}{m}\nabla\chi\Psi^*\Psi + \frac{e}{m}\mathbf{A}\Psi^*\Psi + \frac{e}{m}\nabla\chi\Psi^*\Psi \\
&= \frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right) + \frac{e}{m}\mathbf{A}\Psi^*\Psi = \mathbf{j}
\end{aligned}$$

Ugly, but in the end it wasn't too bad.

(d) [9] Show that with this definition of the current j, conservation of probability works.

We start where we left off in part (b), which was

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\left[\frac{1}{2m}\left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right) + \frac{e}{m}\Psi^*\mathbf{A}\Psi\right] = 0,$$

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0.$$

This was easy only because we did all the hard work in part (b).