Physics 741 – Graduate Quantum Mechanics 1

Solution Set T

1. [20] Suppose that U and V are vector operators with respect to some angular momentum operator J; that is,

$$\left[J_i,U_j\right]=\sum_k i\hbar\varepsilon_{ijk}U_k\ ,\qquad\text{and}\qquad \left[J_i,V_j\right]=\sum_k i\hbar\varepsilon_{ijk}V_k$$

Show that

(a) [4] $S=\mathbf{U}\cdot\mathbf{V}$ is a scalar operator.

We need to prove that S commutes with **J**.

$$\left[J_i,\mathbf{U}\cdot\mathbf{V}\right]=\sum_j\left[J_i,U_jV_j\right]=\sum_j\left\{\left[J_i,U_j\right]V_j+U_j\left[J_i,V_j\right]\right\}=i\hbar\sum_{j,k}\left\{\varepsilon_{ijk}U_kV_j+\varepsilon_{ijk}U_jV_k\right\}$$

Now, rename the dummy indices *j* and *k* as *k* and *j* in the last term, and then use the fact that ε_{ijk} is anti-symmetric under the interchange of its last two indices.

$$\left[J_i,\mathbf{U}\cdot\mathbf{V}\right]=i\hbar\sum_{j,k}U_kV_j\left\{\varepsilon_{ijk}+\varepsilon_{ikj}\right\}=0$$

(b) [18] $\mathbf{W}=\mathbf{U}\times\mathbf{V}$ is a vector operator

Although this can be done with the Levi-Civita symbol, it isn't that much harder just to work out all nine cases by hand.

$$\left[J_x,W_x\right]=\left[J_x,U_yV_z-U_zV_y\right]=U_y\left[J_x,V_z\right]+\left[J_x,U_y\right]V_z-U_z\left[J_x,V_y\right]-\left[J_x,U_z\right]V_y$$
$$=i\hbar\left\{-U_yV_y+U_zV_z-U_zV_z+U_yV_y\right\}=0,$$
$$\left[J_x,W_y\right]=\left[J_x,U_zV_x-U_xV_z\right]=\left[J_x,U_z\right]V_x-U_x\left[J_x,V_z\right]=i\hbar\left\{-U_yV_x+U_xV_y\right\}=i\hbar W_z,$$
$$\left[J_x,W_z\right]=\left[J_x,U_xV_y-U_yV_x\right]=U_x\left[J_x,V_y\right]-\left[J_x,U_y\right]V_x=i\hbar\left\{U_xV_z-U_zV_x\right\}=-i\hbar W_y,$$
$$\left[J_y,W_x\right]=\left[J_y,U_yV_z-U_zV_y\right]=U_y\left[J_y,V_z\right]-\left[J_y,U_z\right]V_y=i\hbar\left\{U_yV_x-U_xV_y\right\}=-i\hbar W_z,$$
$$\left[J_y,W_y\right]=\left[J_y,U_zV_x-U_xV_z\right]=U_z\left[J_y,V_x\right]+\left[J_y,U_z\right]V_x-U_x\left[J_y,V_z\right]-\left[J_y,U_x\right]V_z$$
$$=i\hbar\left\{-U_zV_z+U_xV_x-U_xV_x+U_zV_z\right\}=0,$$
$$\left[J_y,W_z\right]=\left[J_y,U_xV_y-U_yV_x\right]=\left[J_y,U_x\right]V_y-U_y\left[J_y,V_x\right]=i\hbar\left\{-U_zV_y+U_yV_z\right\}=i\hbar W_x,$$
$$\left[J_z,W_x\right]=\left[J_z,U_yV_z-U_zV_y\right]=\left[J_z,U_y\right]V_z-U_z\left[J_z,V_y\right]=i\hbar\left\{-U_xV_z+U_zV_x\right\}=-i\hbar W_y,$$
$$\left[J_z,W_y\right]=\left[J_z,U_zV_x-U_xV_z\right]=U_z\left[J_z,V_x\right]-\left[J_z,U_x\right]V_z=i\hbar\left\{U_zV_y-U_yV_z\right\}=i\hbar W_x,$$
$$\left[J_z,W_z\right]=\left[J_z,U_xV_y-U_yV_x\right]=U_x\left[J_z,V_y\right]+\left[J_z,U_x\right]V_y-U_y\left[J_z,V_x\right]-\left[J_z,U_y\right]V_x$$
$$=i\hbar\left\{-U_xV_x+U_yV_y-U_yV_y+U_xV_x\right\}=0.$$

2. [10] Suppose that U is a vector operator with respect to some angular momentum operator J; that is,

$$\left[J_i, U_j\right] = \sum_k i\hbar \varepsilon_{ijk} U_k \ .$$

Define the operators U_q **as**

$$U_0 = U_z \,, \qquad U_{\pm 1} = \mp\sqrt{\tfrac{1}{2}} U_x - i\sqrt{\tfrac{1}{2}} U_y$$

Show explicitly that this is a spherical tensor with *k* = 1, so that

$$\left[J_z, U_q\right] = \hbar q U_q \quad \text{and} \quad \left[J_\pm, U_q\right] = \hbar\sqrt{2 - q^2 \mp q}\, U_{q\pm 1}$$

We'll simply work them all out

$$\left[J_z, U_0\right] = \left[J_z, U_z\right] = 0,$$

$$\left[J_z, U_{\pm 1}\right] = \sqrt{\tfrac{1}{2}}\left[J_z, \mp U_x - iU_y\right] = i\hbar\sqrt{\tfrac{1}{2}}\left(\mp U_y + iU_x\right) = \pm\hbar\sqrt{\tfrac{1}{2}}\left(\mp U_x - iU_y\right) = \hbar U_{\pm 1}$$

$$\left[J_\pm, U_0\right] = \left[J_x \pm iJ_y, U_z\right] = i\hbar\left(-U_y \pm iU_x\right) = \hbar\left(\mp U_x - iU_y\right) = \hbar\sqrt{2} U_{\pm 1}$$

$$\left[J_\pm, U_{\pm 1}\right] = \sqrt{\tfrac{1}{2}}\left[J_x \pm iJ_y, \mp U_x - iU_y\right] = \sqrt{\tfrac{1}{2}}\left\{-i\left[J_x, U_y\right] - i\left[J_y, U_x\right]\right\} = \hbar\sqrt{\tfrac{1}{2}}\left\{U_z - U_z\right\} = 0,$$

$$\left[J_\pm, U_{\mp 1}\right] = \sqrt{\tfrac{1}{2}}\left[J_x \pm iJ_y, \pm U_x - iU_y\right] = \sqrt{\tfrac{1}{2}}\left\{-i\left[J_x, U_y\right] + i\left[J_y, U_x\right]\right\} = \hbar\sqrt{\tfrac{1}{2}}\left\{U_z + U_z\right\}$$
$$= \hbar\sqrt{2} U_0 .$$

They all worked out as they should.