Physics 741 – Graduate Quantum Mechanics 1

Solution Set R

1. [10] Suppose we have a particle with spin ½. Let's define the spin operator S_θ as

$$S_\theta = \tfrac{1}{2}\hbar\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right)$$

In other words, we are measuring the spin of the particle along an axis that is at an angle θ compared to the *z*-axis

(a) [5] Verify that

$$\left|+_\theta\right\rangle = \cos\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle$$
$$\left|-_\theta\right\rangle = -\sin\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle$$

are normalized eigenvectors of S_θ, and determine the eigenvalues. In other words, you have to demonstrate 3 things, (i) they are normalized, (ii) they are eigenvectors, (iii) determine their eigenvalues.

To check that they are normalized, we see that

$$\begin{aligned}
\left\langle +_\theta \middle| +_\theta \right\rangle &= \left[\cos\left(\tfrac{1}{2}\theta\right)\left\langle +_z\right| + \sin\left(\tfrac{1}{2}\theta\right)\left\langle -_z\right|\right]\left[\cos\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle\right] \\
&= \cos^2\left(\tfrac{1}{2}\theta\right) + \sin^2\left(\tfrac{1}{2}\theta\right) = 1 \\
\left\langle -_\theta \middle| -_\theta \right\rangle &= \left[-\sin\left(\tfrac{1}{2}\theta\right)\left\langle +_z\right| + \cos\left(\tfrac{1}{2}\theta\right)\left\langle -_z\right|\right]\left[-\sin\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle\right] \\
&= \sin^2\left(\tfrac{1}{2}\theta\right) + \cos^2\left(\tfrac{1}{2}\theta\right) = 1
\end{aligned}$$

To check that they are eigenstates, we see that

$$\begin{aligned}
S_\theta\left|+_\theta\right\rangle &= \tfrac{1}{2}\hbar\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right)\left[\cos\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle\right] \\
&= \tfrac{1}{2}\hbar\begin{bmatrix}\cos\theta\cos\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle - \cos\theta\sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle \\ +\sin\theta\cos\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle + \sin\theta\sin\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle\end{bmatrix} \\
&= \tfrac{1}{2}\hbar\left\{\left[\cos\theta\cos\left(\tfrac{1}{2}\theta\right) + \sin\theta\sin\left(\tfrac{1}{2}\theta\right)\right]\left|+_z\right\rangle + \left[\sin\theta\cos\left(\tfrac{1}{2}\theta\right) - \cos\theta\sin\left(\tfrac{1}{2}\theta\right)\right]\left|-_z\right\rangle\right\} \\
&= \tfrac{1}{2}\hbar\cos\left(\theta - \tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \tfrac{1}{2}\hbar\sin\left(\theta - \tfrac{1}{2}\theta\right)\left|-_z\right\rangle = +\tfrac{1}{2}\hbar\left|+_\theta\right\rangle, \\
S_\theta\left|-_\theta\right\rangle &= \tfrac{1}{2}\hbar\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right)\left[-\sin\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle\right] \\
&= \tfrac{1}{2}\hbar\begin{bmatrix}-\cos\theta\sin\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle - \cos\theta\sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle \\ -\sin\theta\sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle + \sin\theta\cos\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle\end{bmatrix} \\
&= \tfrac{1}{2}\hbar\left\{\left[\sin\theta\cos\left(\tfrac{1}{2}\theta\right) - \cos\theta\sin\left(\tfrac{1}{2}\theta\right)\right]\left|+_z\right\rangle - \left[\sin\theta\sin\left(\tfrac{1}{2}\theta\right) + \cos\theta\cos\left(\tfrac{1}{2}\theta\right)\right]\left|-_z\right\rangle\right\} \\
&= \tfrac{1}{2}\hbar\sin\left(\theta - \tfrac{1}{2}\theta\right)\left|+_z\right\rangle - \tfrac{1}{2}\hbar\cos\left(\theta - \tfrac{1}{2}\theta\right)\left|-_z\right\rangle = -\tfrac{1}{2}\hbar\left|-_\theta\right\rangle,
\end{aligned}$$

The eigenvalues are obviously $\pm\tfrac{1}{2}\hbar$.

(b) [3] Suppose a particle is initially in the state $|+_z\rangle$. If a subsequent measurement at angle θ is done, what is the probability the result will come out positive? What is the state immediately after the measurement is done?

According to our rules for probability, we square the overlap of our quantum state with the eigenstate with a positive eigenvalue, which gives us

$$P\left(\tfrac{1}{2}\hbar\right) = \left|\langle +_\theta|\psi\rangle\right|^2 = \left|\langle +_\theta|+_z\rangle\right|^2 = \left|\cos\left(\tfrac{1}{2}\theta\right)\langle +_z|+_z\rangle + \sin\left(\tfrac{1}{2}\theta\right)\langle -_z|+_z\rangle\right|^2 = \cos^2\left(\tfrac{1}{2}\theta\right)$$

The quantum state afterwards is then given by

$$\psi\left(t^+\right) = \frac{|+_\theta\rangle\langle +_\theta|+_z\rangle}{\sqrt{P\left(+\tfrac{1}{2}\hbar\right)}} = \frac{|+_\theta\rangle\cos\left(\tfrac{1}{2}\theta\right)}{\sqrt{\cos^2\left(\tfrac{1}{2}\theta\right)}} = |+_\theta\rangle$$

In this case, we didn't actually need to do the computation, since there is only one state with this eigenvalue.

(c) [2] After the measurement at angle θ yields a positive result, *another* measurement is done, this time at angle θ' is performed. What is the probability this time that the result comes out positive?

The result is simply

$$P\left(+\tfrac{1}{2}\hbar\right) = \left|\langle +_{\theta'}|+_\theta\rangle\right|^2 = \left|\left[\cos\left(\tfrac{1}{2}\theta'\right)\langle +_z| + \sin\left(\tfrac{1}{2}\theta'\right)\langle -_z|\right]\left[\cos\left(\tfrac{1}{2}\theta\right)|+_z\rangle + \sin\left(\tfrac{1}{2}\theta\right)|-_z\rangle\right]\right|^2$$
$$= \left|\cos\left(\tfrac{1}{2}\theta'\right)\cos\left(\tfrac{1}{2}\theta\right) + \sin\left(\tfrac{1}{2}\theta'\right)\sin\left(\tfrac{1}{2}\theta\right)\right|^2 = \cos^2\left[\tfrac{1}{2}\left(\theta' - \theta\right)\right]$$

2. [10] It is common to have to calculate matrix elements of the form

$$\langle n', l', m_l', m_s'|\mathbf{L}\cdot\mathbf{S}|n, l, m_l, m_s\rangle,$$

where L and S are the orbital and spin angular momenta respectively, and *l*, m_l, and m_s are quantum numbers corresponding to the operators L^2, L_z, and S_z, respectively (*n* represents some sort of radial quantum number).

(a) [2] Show that $\mathbf{L}\cdot\mathbf{S}$ can be written in a simple way in terms of L^2, S^2, and $\mathbf{J}^2 = \left(\mathbf{L}+\mathbf{S}\right)^2$. You may assume any commutation relations that you know are true about L and S, or that you proved in a previous problem set.

It is easy to see that

$$\mathbf{J}^2 = \left(\mathbf{L}+\mathbf{S}\right)^2 = \mathbf{L}^2 + 2\mathbf{L}\cdot\mathbf{S} + \mathbf{S}^2$$

Where we recall that **L** and **S** commute. It is easy then to rewrite this as

$$\mathbf{L}\cdot\mathbf{S} = \tfrac{1}{2}\left(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2\right)$$

(b) [2] Show that the operator $\mathbf{L}\cdot\mathbf{S}$ commutes with $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$.

This is trivial. We already know that $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$ all commute with each other, so any linear combination of them must commute as well.

(c) [3] A more intelligent basis to use would be eigenstates of $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$, and J_z, so our states would look like $\left|n,l,j,m_j\right\rangle$ (the constant *s* is implied). Assuming our states are orthonormal, work out a simple formula for

$$\left\langle n',l',j',m_j'\right|\mathbf{L}\cdot\mathbf{S}\left|n,l,j,m_j\right\rangle$$

Again, this is trivial. We have

$$\begin{aligned}\left\langle n',l',j',m_j'\right|\mathbf{L}\cdot\mathbf{S}\left|n,l,j,m_j\right\rangle &= \tfrac{1}{2}\left\langle n',l',j',m_j'\right|\left(\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2\right)\left|n,l,j,m_j\right\rangle \\ &= \tfrac{1}{2}\hbar^2\left(j^2+j-l^2-l-s^2-s\right)\left\langle n',l',j',m_j' \middle| n,l,j,m_j\right\rangle \\ &= \tfrac{1}{2}\hbar^2\left(j^2+j-l^2-l-s^2-s\right)\delta_{n'n}\delta_{l'l}\delta_{j'j}\delta_{m_j'm_j}\end{aligned}$$

(d) [3] For arbitrary *l* =0, 1, 2, … and *s* = ½, what are the possible values of *j*? Work out all matrix elements of the form above in this case.

The quantum number *j* runs from $j=|l-s|=\left|l-\tfrac{1}{2}\right|$ to $j=l+s=l+\tfrac{1}{2}$. This means that the only allowed values are $j=l\pm\tfrac{1}{2}$ for $l>0$, and for $l=0$ only $j=\tfrac{1}{2}$ is allowed. Setting $j=l\pm\tfrac{1}{2}$, we have

$$\begin{aligned}\left(j^2+j-l^2-l-s^2-s\right) &= \left(l\pm\tfrac{1}{2}\right)^2+\left(l\pm\tfrac{1}{2}\right)-l^2-l-\left(\tfrac{1}{2}\right)^2-\tfrac{1}{2} = l^2\pm l+\tfrac{1}{4}+l\pm\tfrac{1}{2}-l^2-l-\tfrac{3}{4} \\ &= \pm l\pm\tfrac{1}{2}-\tfrac{1}{2}\end{aligned}$$

so

$$\left\langle n',l',j',m_j'\right|\mathbf{L}\cdot\mathbf{S}\left|n,l,j,m_j\right\rangle = \begin{cases}\tfrac{1}{2}\hbar^2 l\,\delta_{n'n}\delta_{l'l}\delta_{j'j}\delta_{m_j'm_j} & \text{if } j=l+\tfrac{1}{2} \\ \tfrac{1}{2}\hbar^2\left(-l-1\right)\delta_{n'n}\delta_{l'l}\delta_{j'j}\delta_{m_j'm_j} & \text{if } j=l-\tfrac{1}{2}\end{cases}$$