Physics 741 – Graduate Quantum Mechanics 1

# Solution Set Q

1. **[10] In chapter nine, section A, we were searching for matrices $D(\mathcal{R})$ which satisfy**

$$D(\mathcal{R}_1)D(\mathcal{R}_2) = D(\mathcal{R}_1\mathcal{R}_2)$$

**One easy way to make this equation work out is to define**

$$D(\mathcal{R}) = \mathcal{R}$$

**Our goal in this problem is to identify the spin.**

**(a) [4] Using the equations (7.26) and the definition of the spin matrices (9.6), work out the three spin matrices S.**

Equations (7.26) give the rotation matrices around each of the three axes for arbitrary angle $\theta$. If we write these to order $\theta$, we see that they give

$$\mathcal{R}(\hat{\mathbf{i}},\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{pmatrix}$$

$$\mathcal{R}(\hat{\mathbf{j}},\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{pmatrix}$$

$$\mathcal{R}(\hat{\mathbf{k}},\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We now match this to the formula (9.6), which says to linear order

$$\mathcal{R}(\hat{\mathbf{r}},\theta) = 1 - i\theta\hat{\mathbf{r}}\cdot\mathbf{S}/\hbar \quad \text{so} \quad \hat{\mathbf{r}}\cdot\mathbf{S} = i\hbar\left[\mathcal{R}(\hat{\mathbf{r}},\theta) - 1\right]$$

We can now read off the three spin matrices pretty easily.

$$S_x = \hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \hbar\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \hbar\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**(b) [3] Find the eigenvalues of $S_z$. This should be enough for you to conjecture what value of *s* this representation.**

The grammar of the problem leaves something to be desired, but the easiest way to find these is to use the characteristic equation, which is

$$0 = \det\left(S_z - \lambda\mathbf{1}\right) = \det\begin{pmatrix} -\lambda & -i\hbar & 0 \\ i\hbar & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = -\lambda^3 + \hbar^2\lambda = -\lambda\left(\lambda - \hbar\right)\left(\lambda + \hbar\right)$$

The three solutions of this are $\lambda = 0, \hbar, -\hbar$, which are the three values we would expect for spin 1. So we suspect $s = 1$. We don't recognize it in this form, because we have not written it in the basis where $S_z$ is diagonalized.

**(c) [3] Check explicitly that $\mathbf{S}^2$ is a constant matrix with the appropriate value.**

$$\mathbf{S}^2 = \hbar^2\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2\right]$$

$$= \hbar^2\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \hbar^2\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The result is supposed to be $\mathbf{S}^2 = \left(s^2 + s\right)\hbar^2 = 2\hbar^2$, so it worked out.

**2. [10] Suppose that L and S are two sets of commuting angular momentum-like operators, so that**

$$\begin{aligned} &\left[L_x, L_y\right] = i\hbar L_z, \quad \left[L_y, L_z\right] = i\hbar L_x, \quad \left[L_z, L_x\right] = i\hbar L_y, \\ &\left[S_x, S_y\right] = i\hbar S_z, \quad \left[S_y, S_z\right] = i\hbar S_x, \quad \left[S_z, S_x\right] = i\hbar S_y, \end{aligned} \qquad \text{and} \qquad \left[L_i, S_j\right] = 0.$$

**In this problem, you may assume from the commutation relations that it follows that**

$$\left[\mathbf{L}^2, \mathbf{L}\right] = 0 = \left[\mathbf{S}^2, \mathbf{S}\right]$$

**(a) [2] Define $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Show that $\mathbf{J}$ is also an angular momentum-like operator. It follows automatically that $\left[\mathbf{J}^2, \mathbf{J}\right] = 0$.**

We simply have to work out the commutator of each component of **J** with each other. We'll take advantage of the Levi-Civita tensor to show that

$$\left[J_i, J_j\right] = \left[L_i + S_i, L_j + S_j\right] = \left[L_i, L_j\right] + \left[S_i, S_j\right] = i\hbar\sum_k \varepsilon_{ijk} L_k + i\hbar\sum_k \varepsilon_{ijk} S_k = i\hbar\sum_k \varepsilon_{ijk} J_k$$

This saved us the work of doing it three times.

**(b) [2] Show that $\left[\mathbf{L}^2, \mathbf{J}\right] = 0 = \left[\mathbf{S}^2, \mathbf{J}\right]$.**

This is really pretty trivial.

$$\left[\mathbf{L}^2, \mathbf{J}\right] = \left[\mathbf{L}^2, \mathbf{L} + \mathbf{S}\right] = \left[\mathbf{L}^2, \mathbf{L}\right] + \left[\mathbf{L}^2, \mathbf{S}\right] = 0 + 0 = 0,$$
$$\left[\mathbf{S}^2, \mathbf{J}\right] = \left[\mathbf{S}^2, \mathbf{L} + \mathbf{S}\right] = \left[\mathbf{S}^2, \mathbf{L}\right] + \left[\mathbf{S}^2, \mathbf{S}\right] = 0 + 0 = 0.$$

**(c) [3] Convince yourself (and me) that the four operators $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$, and $J_z$ all commute with each other (this is six commutators in all).**

Since $\mathbf{L}^2$ and $\mathbf{S}^2$ commute with **J**, it follows automatically that they commute with $\mathbf{J}^2$ and $J_z$ (that's four commutators so far). Since all the **L**'s and **S**'s commute with each other, it follows that $\mathbf{L}^2$ and $\mathbf{S}^2$ commute with each other. Finally, as mentioned in part (a), since $\left[\mathbf{J}^2, \mathbf{J}\right] = 0$, $\mathbf{J}^2$ commutes with $J_z$.

**(d) [3] Convince yourself that $L_z$ and $S_z$ do <u>not</u> commute with $\mathbf{J}^2$.**

We simply try to do the commutation relations and see if it works.

$$\left[\mathbf{J}^2, L_z\right] = \left[\left(\mathbf{L} + \mathbf{S}\right)^2, L_z\right] = \left[\mathbf{L}^2 + 2\mathbf{L}\cdot\mathbf{S} + \mathbf{S}^2, L_z\right] = \left[\mathbf{L}^2, L_z\right] + 2\left[\mathbf{L}\cdot\mathbf{S}, L_z\right] + \left[\mathbf{S}^2, L_z\right]$$
$$= 2\left[L_x, L_z\right]S_x + 2\left[L_y, L_z\right]S_y + 2\left[L_z, L_z\right]S_z = -2i\hbar L_y S_x + 2i\hbar L_x S_y$$
$$\left[\mathbf{J}^2, S_z\right] = \left[\left(\mathbf{L} + \mathbf{S}\right)^2, S_z\right] = \left[\mathbf{L}^2 + 2\mathbf{L}\cdot\mathbf{S} + \mathbf{S}^2, S_z\right] = \left[\mathbf{L}^2, S_z\right] + 2\left[\mathbf{L}\cdot\mathbf{S}, S_z\right] + \left[\mathbf{S}^2, S_z\right]$$
$$= 2L_x\left[S_x, S_z\right] + 2L_y\left[S_y, S_z\right] + 2L_z\left[S_z, S_z\right] = -2i\hbar L_x S_y + 2i\hbar L_y S_x$$

Not surprisingly, the sum of these two expressions is zero, which follows from $\left[\mathbf{J}^2, \mathbf{J}\right] = 0$.