Physics 741 – Graduate Quantum Mechanics 1

Solution Set N

1. [10] Let $\mathbf{J}=\left(J_x,J_y,J_z\right)$ be three Hermitian operators that commute with the Hamiltonian *H*, and have angular momentum-like commutation relations

$$\left[J_x,J_y\right]=i\hbar J_z,\qquad \left[J_y,J_z\right]=i\hbar J_x,\qquad \left[J_z,J_x\right]=i\hbar J_y.$$

Define the operators

$$\mathbf{J}^2=J_x^2+J_y^2+J_z^2\qquad\text{and}\qquad J_\pm=J_x\pm iJ_y$$

Show each of the following is true:

(a) [6] $\left[\mathbf{J}^2,\mathbf{J}\right]=0$ (this is three identities), and therefore $\left[J^2,J_\pm\right]=0$

We simply begin working them out:

$$\begin{aligned}\left[J^2,J_x\right]&=0+\left[J_y^2,J_x\right]+\left[J_z^2,J_x\right]=J_y\left[J_y,J_x\right]+\left[J_y,J_x\right]J_y+J_z\left[J_z,J_x\right]+\left[J_z,J_x\right]J_z\\&=i\hbar\left(-J_yJ_z-J_zJ_y+J_zJ_y+J_yJ_z\right)=0,\\\left[J^2,J_y\right]&=\left[J_x^2,J_y\right]+0+\left[J_z^2,J_y\right]=J_x\left[J_x,J_y\right]+\left[J_x,J_y\right]J_x+J_z\left[J_z,J_y\right]+\left[J_z,J_y\right]J_z\\&=i\hbar\left(J_xJ_z+J_zJ_x-J_zJ_x-J_xJ_z\right)=0,\\\left[J^2,J_z\right]&=\left[J_x^2,J_z\right]+\left[J_y^2,J_z\right]+0=J_x\left[J_x,J_z\right]+\left[J_x,J_z\right]J_x+J_y\left[J_y,J_z\right]+\left[J_y,J_z\right]J_y\\&=i\hbar\left(-J_xJ_y-J_yJ_x+J_yJ_x+J_xJ_y\right)=0.\end{aligned}$$

(b) [2] $\left[J_z,J_\pm\right]=\pm\hbar J_\pm$

Again, this is most easily done by just working it out:

$$\left[J_z,J_\pm\right]=\left[J_z,J_x\right]\pm i\left[J_z,J_y\right]=i\hbar J_y\mp i^2\hbar J_x=\pm\hbar J_x+i\hbar J_y=\pm\hbar\left(J_x\pm i\hbar J_y\right)=\pm\hbar J_\pm.$$

(c) [3] $\mathbf{J}^2=J_\mp J_\pm+J_z^2\pm\hbar J_z$

It is easiest to expand the right side and show that it is equal to the left side.

$$\begin{aligned}J_\mp J_\pm+J_z^2\pm\hbar J_z&=\left(J_x\mp iJ_y\right)\left(J_x\pm iJ_y\right)+J_z^2\pm\hbar J_z=J_x^2\pm iJ_xJ_y\mp iJ_yJ_x+J_y^2+J_z^2\pm\hbar J_z\\&=J_x^2+J_y^2+J_z^2\pm i\left[J_x,J_y\right]\pm\hbar J_z=J_x^2+J_y^2+J_z^2\pm i^2\hbar J_z\pm\hbar J_z=J^2\end{aligned}$$

2. [10] For *j* = 2, we will work out the explicit form for all of the matrices $\mathbf{J}$.

(a) [5] Write out the expression for J_z and $J_\pm$ as an appropriately sized matrix.

Since $j = 2$, the matrix will be of size $2 \cdot 2 + 1 = 5$. For J_3, we will have a diagonal matrix with elements running from $2\hbar$ down to $-2\hbar$. For J_+, we will have elements just along the diagonal, where the value in the row labeled by m will be $\hbar\sqrt{j^2 + j - m^2 + m}$, and J_- is just the Hermitian conjugate of J_+. Hence we have

$$J_z = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad J_+ = \hbar \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

(b) [2] Write out J_x and J_y.

This is just a matter of taking $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/2i$

$$J_x = \hbar \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & \sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_y = \hbar \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{3/2} & 0 & 0 \\ 0 & i\sqrt{3/2} & 0 & -i\sqrt{3/2} & 0 \\ 0 & 0 & i\sqrt{3/2} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}.$$

(c) [3] Check explicitly that $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ is a constant matrix with the appropriate value.

$$J^2 = \hbar^2 \begin{pmatrix} 1 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 5/2 & 0 & 3/2 & 0 \\ \sqrt{3/2} & 0 & 3 & 0 & \sqrt{3/2} \\ 0 & 3/2 & 0 & 5/2 & 0 \\ 0 & 0 & \sqrt{3/2} & 0 & 1 \end{pmatrix} + \hbar^2 \begin{pmatrix} 1 & 0 & -\sqrt{3/2} & 0 & 0 \\ 0 & 5/2 & 0 & -3/2 & 0 \\ -\sqrt{3/2} & 0 & 3 & 0 & -\sqrt{3/2} \\ 0 & -3/2 & 0 & 5/2 & 0 \\ 0 & 0 & -\sqrt{3/2} & 0 & 1 \end{pmatrix}$$

$$+\hbar^2 \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = \hbar^2 \begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} = 6\hbar^2 \mathbf{1}$$

The appropriate value is $\hbar^2 \left(j^2 + j \right) = 6\hbar^2$.

3. [5] The Pauli matrices are defined in equation (8.30). Using the Pauli matrices, show that

(a) [2] $(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^2 = 1$ for any unit vector $\hat{\mathbf{r}}$

Let $\hat{\mathbf{r}} = (x, y, z)$, with $x^2 + y^2 + z^2 = 1$. Then

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^2 = \left(x\sigma_x + y\sigma_y + z\sigma_z \right)^2$$
$$= \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} = \begin{pmatrix} z^2 + x^2 + y^2 & 0 \\ 0 & x^2 + y^2 + z^2 \end{pmatrix} = 1$$

(b) [3] $\exp\left(-\tfrac{1}{2} i\theta \boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\right) = \cos\left(\tfrac{1}{2}\theta\right) - i \sin\left(\tfrac{1}{2}\theta\right)(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})$

$$\exp\left(-\tfrac{1}{2} i\theta \boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\tfrac{1}{2} i\theta \boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\right)^n = \sum_{n \text{ even}} \frac{1}{n!} \left(-\tfrac{1}{2} i\theta\right)^n (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^n + \sum_{n \text{ odd}} \frac{1}{n!} \left(-\tfrac{1}{2} i\theta\right)^n (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})^n$$
$$= \left[1 - \tfrac{1}{2!}\left(-\tfrac{1}{2}\theta\right)^2 + \tfrac{1}{4!}\left(-\tfrac{1}{2}\theta\right)^4 - + \cdots \right] \cdot 1$$
$$+ \left[i\left(-\tfrac{1}{2}\theta\right) - \tfrac{i}{3!}\left(-\tfrac{1}{2}\theta\right)^3 + \tfrac{i}{5!}\left(-\tfrac{1}{2}\theta\right)^5 - + \cdots \right] (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})$$
$$= \cos\left(-\tfrac{1}{2}\theta\right) + i \sin\left(-\tfrac{1}{2}\theta\right)(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) = \cos\left(\tfrac{1}{2}\theta\right) - i \sin\left(\tfrac{1}{2}\theta\right)(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}})$$