Physics 741 – Graduate Quantum Mechanics 1

# Solution Set G

**1. [5] Prove the parity operator $\Pi$, defined by $\Pi\psi(\mathbf{r}) = \psi(-\mathbf{r})$ is both Hermitian and unitary.**

To show it is Hermitian, we must show that $\langle\phi|\Pi|\psi\rangle^* = \langle\psi|\Pi|\phi\rangle$, so

$$\langle\phi|\Pi|\psi\rangle^* = \left[\iiint d^3\mathbf{r}\,\phi^*(\mathbf{r})\psi(-\mathbf{r})\right]^* = \iiint d^3\mathbf{r}\,\psi^*(-\mathbf{r})\phi(\mathbf{r}) = \iiint d^3\mathbf{r}\,\psi^*(\mathbf{r})\phi(-\mathbf{r})$$
$$= \langle\psi|\Pi|\phi\rangle$$

Now that we know it is Hermitian, we can take advantage of this to show that

$$\Pi^\dagger\Pi\psi(\mathbf{r}) = \Pi^2\psi(\mathbf{r}) = \Pi\psi(-\mathbf{r}) = \psi(\mathbf{r})$$

Since this is true for all wave functions, it follows that $\Pi^\dagger\Pi = 1$.

**2. [15] Consider the Hermitian matrix**

$$H = E_0\begin{pmatrix} 0 & 3i & 0 \\ -3i & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

**(a) [10] Find all three eigenvalues and eigenvectors of $H$**

We note first that it is block-diagonal, as I have sketched in with dashed lines in the problem itself, reducing the matrix to a 2 × 2 matrix and a trivial 1 × 1 matrix:

$$H_2 = E_0\begin{pmatrix} 0 & 3i \\ -3i & 8 \end{pmatrix} \quad \text{and} \quad H_1 = E_0(8)$$

The matrix $H_1$ has eigenvalue $8E_0$, and eigenvector (1), which makes it trivial. The eigenvalues of matrix $H_2$ can be found using the characteristic equation

$$0 = \det(H_2 - \lambda\mathbf{1}) = \begin{pmatrix} -\lambda & 3iE_0 \\ -3iE_0 & 8E_0 - \lambda \end{pmatrix} = \lambda^2 - 8\lambda E_0 - (3i)(-3i)E_0^2 = \lambda^2 - 8\lambda E_0 - 9E_0^2$$
$$= (\lambda - 9E_0)(\lambda + E_0)$$

This has solutions $\lambda = 9E_0$ and $\lambda = -E_0$. To find each of these values, we put in an arbitrary vector and solve the eigenvalue equation. For example, for $\lambda = 9E_0$, we have

$$\begin{pmatrix} 0 & 3iE_0 \\ -3iE_0 & 8E_0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 9E_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$
$$\begin{pmatrix} 3i\beta \\ -3i\alpha + 8\beta \end{pmatrix} = \begin{pmatrix} 9\alpha \\ 9\beta \end{pmatrix}$$

The first of these equations implies $\beta = -3i\alpha$; if we plug this into the second, we find that it is also automatically satisfied. We also want the eigenvector normalized, so

$$1 = |\alpha|^2 + |\beta|^2 = 10|\alpha|^2$$

We have an arbitrary phase to choose; if we pick $\alpha$ to be real and positive, $\alpha = 1/\sqrt{10}$, and we have the eigenvector

$$|9E_0\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3i \end{pmatrix}$$

For the other eivenvector, we have

$$\begin{pmatrix} 0 & 3iE_0 \\ -3iE_0 & 8E_0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -E_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$
$$\begin{pmatrix} 3i\beta \\ -3i\alpha + 8\beta \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\beta \end{pmatrix}$$

Both equations imply $\beta = i\alpha/3$. Our normalization condition becomes

$$1 = |\alpha|^2 + |\beta|^2 = \tfrac{10}{9}|\alpha|^2$$

Once again we pick $\alpha$ to be real and positive, $\alpha = 1/\sqrt{10}$, and we have the eigenvector

$$|-E_0\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ i \end{pmatrix}$$

Returning to the full three-dimensional space, our eigenvectors are

$$|-E_0\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ i \\ 0 \end{pmatrix}, \quad |9E_0\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3i \\ 0 \end{pmatrix}, \quad |8E_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Your answers might be slightly different, in that the phases could be different, or the eigenvectors could be listed in a different order.

**(b) [5] Construct the unitary matrix *V* which diagonalizes *H*. Check explicitly that $V^\dagger V = 1$ and $V^\dagger H V = H'$ is real and diagonal.**

The unitary matrix *V* just consists of the eigenvectors listed in any order, so we have

$$V = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{i}{\sqrt{10}} & -\frac{3i}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Your answer could be different, in that the columns could come in a different order, and each column could be multiplied by an arbitrary phase.

We have ahead of us some boring matrix multiplication.

$$V^\dagger V = \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{i}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{3i}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{i}{\sqrt{10}} & -\frac{3i}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{9}{10}+\frac{1}{10} & \frac{3}{10}-\frac{3}{10} & 0 \\ \frac{3}{10}-\frac{3}{10} & \frac{1}{10}+\frac{9}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$V^\dagger H V = E_0 \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{i}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{3i}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3i & 0 \\ -3i & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{i}{\sqrt{10}} & -\frac{3i}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= E_0 \begin{pmatrix} \frac{3}{\sqrt{10}} & -\frac{i}{\sqrt{10}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{3i}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{\sqrt{10}} & \frac{9}{\sqrt{10}} & 0 \\ -\frac{i}{\sqrt{10}} & -\frac{27i}{\sqrt{10}} & 0 \\ 0 & 0 & 8 \end{pmatrix} = E_0 \begin{pmatrix} -\frac{9}{10}-\frac{1}{10} & \frac{27}{10}-\frac{27}{10} & 0 \\ -\frac{3}{10}+\frac{3}{10} & \frac{9}{10}+\frac{81}{10} & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= E_0 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

As you can see, $V^\dagger V = 1$ and $V^\dagger H V$ is real and diagonal (and has the eigenvalues on its diagonal).