Physics 741 – Graduate Quantum Mechanics 1

Solution Set E

1. [15] A particle of mass *m* and energy *E* scatters from the negative-*x* direction off of a delta-function potential:

$$V(x) = \lambda\delta(x).$$

(a) [4] For the regions *x* < 0 and *x* > 0, find general equations for the wave, eliminating any terms that are physically inappropriate.

In both regions, the potential vanishes, and therefore the solutions just look like $e^{\pm ikx}$. On the right side, however, we want only a transmitted wave, so we throw out one of the solutions and write

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx},$$
$$\psi_{II}(x) = Ce^{ikx},$$
$$E = \hbar^2k^2/2m.$$

(b) [5] Integrate Schrödinger's time-independent equation across the boundary to obtain an equation relating the derivative of the wave function on either side of the boundary. Will the wave function itself be continuous?

As in class, we start with Schrödinger's time independent equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + \lambda\delta(x)\psi(x) = E\psi(x)$$

and integrate it across the boundary at *x* = 0:

$$\int_{-\varepsilon}^{\varepsilon}\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + \lambda\delta(x)\psi(x)\right\}dx = E\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx,$$
$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\psi(x)\bigg|_{-\varepsilon}^{\varepsilon} + \lambda\psi(0) = 0,$$
$$-\frac{\hbar^2}{2m}\psi'_{II}(0) + \frac{\hbar^2}{2m}\psi'_I(0) + \lambda\psi(0) = 0.$$

This implies a finite discontinuity in the derivative, which means that the function itself is presumably continuous, so we also have

$$\psi_{II}(0) = \psi_I(0).$$

(c) [6] Solve the equations and deduce the transmission and reflection coefficients *T* and *R*. Check that *T* + *R* = 1.

The continuity of the wave function tells us that

$$A + B = C\ ,$$

while the first derivative condition tells us that

$$-\frac{\hbar^2}{2m} ikC + \frac{\hbar^2}{2m} ik\left(A - B\right) + \lambda\left(A + B\right) = 0.$$

If we substitute the first equation into the second, this becomes

$$\frac{\hbar^2}{2m} ik\left(A - B\right) + \lambda\left(A + B\right) = \frac{\hbar^2}{2m} ik\left(A + B\right), \quad \text{so that } \lambda\left(A + B\right) = \hbar^2 ikB/m,$$

We can then solve this and find

$$\frac{B}{A} = \frac{\lambda m}{ik\hbar^2 - \lambda m} \qquad \text{and} \qquad \frac{C}{A} = 1 + \frac{B}{A} = \frac{ik\hbar^2}{ik\hbar^2 - \lambda m}$$

The reflection and transmission coefficients are then

$$R = \frac{\left|j_B\right|}{j_A} = \frac{\left|B\right|^2 k}{\left|A\right|^2 k} = \frac{\lambda^2 m^2}{k^2\hbar^4 + \lambda^2 m^2} \qquad \text{and} \qquad T = \frac{j_C}{j_A} = \frac{\left|C\right|^2 k}{\left|A\right|^2 k} = \frac{k^2\hbar^4}{k^2\hbar^4 + \lambda^2 m^2}$$

In this form it is obvious that $R + T = 1$.

2. [15] A particle of mass *m* lies in the one-dimensional infinite square well, which has potential

$$V\left(x\right) = \begin{cases} 0 & \text{if } 0 < x < a \\ \infty & \text{otherwise} \end{cases},$$

(a) [6] Show that for any positive integer *n*, the equation given below satisfies the time-independent Schrödinger equation, and determine the energy *E* and the normalization constant *N*.

$$\psi_n\left(x\right) = \begin{cases} N \sin\left(\dfrac{\pi n x}{a}\right) & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

We simply plug this solution into the time-independent Schrödinger equation to yield

$$E_n \psi_n\left(x\right) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_n\left(x\right)$$

$$E_n N \sin\left(\frac{\pi n x}{a}\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} N \sin\left(\frac{\pi n x}{a}\right) = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2 N \sin\left(\frac{\pi n x}{a}\right)$$

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$$

To find the normalization, we use the normalization condition.

$$1 = \int_{-\infty}^{\infty} \left|\psi_n(x)\right|^2 dx = N^2 \int_0^a \sin^2(\pi n x/a)\, dx = N^2 \left[\tfrac{1}{2} x - \sin(2\pi n x/a)\right]_0^a = \tfrac{1}{2} N^2 a,$$

$$N = \sqrt{2/a}$$

(b) [5] At t = 0, the wave function takes the form

$$\Psi(x, t=0) = \frac{2}{\sqrt{5a}} \sin^3\left(\frac{\pi x}{a}\right)$$

Rewrite this in the form

$$\Psi(x, t=0) = \sum_i c_i \psi_i(x)$$

and determine all the non-vanishing coefficients c_i.

We will take advantage of the identity given below, which we first rewrite as

$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin(3\theta)$$

So we have

$$\Psi(x, t=0) = \frac{2}{\sqrt{5a}}\left[\frac{3}{4}\sin\left(\frac{\pi x}{a}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{a}\right)\right] = \frac{3}{2\sqrt{5a}}\sin\left(\frac{\pi x}{a}\right) - \frac{1}{2\sqrt{5a}}\sin\left(\frac{3\pi x}{a}\right)$$

$$= \frac{3}{2\sqrt{10}}\psi_1(x) - \frac{1}{2\sqrt{10}}\psi_3(x)$$

In other words, we have $c_1 = \frac{3}{2\sqrt{10}}$, $c_3 = -\frac{1}{2\sqrt{10}}$, and the rest of the c_i's vanish. This means I didn't normalize the wave properly, but that doesn't stop us from finishing the problem.

(c) [4] Find the wave function $\Psi(x,t)$ at all later times.

The general solution is

$$\Psi(x,t) = \sum_i c_i \psi_i(x) e^{-iE_i t/\hbar}$$

In this case, we have

$$\Psi(x,t) = \frac{3}{2\sqrt{5a}}\sin\left(\frac{\pi x}{a}\right)\exp\left(-i\frac{\pi^2 \hbar t}{2ma^2}\right) - \frac{1}{2\sqrt{5a}}\sin\left(\frac{3\pi x}{a}\right)\exp\left(-i\frac{9\pi^2 \hbar t}{2ma^2}\right)$$