

Physics 741 – Graduate Quantum Mechanics 1  
Solution Set D

1. [10] One solution of the 2D Harmonic oscillator Schrodinger equation looks like this:

$$\Psi(x, y, t) = (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t}$$

- (a) [3] Find the probability density  $\rho(x, y, t)$  at all times

$$\rho(x, y, t) = \Psi^* \Psi = (x - iy) e^{-A(x^2+y^2)/2} e^{i\omega t} (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t} = (x^2 + y^2) e^{-A(x^2+y^2)}$$

- (b) [4] Find the probability current  $\mathbf{j}(x, y, t)$  at all times.

$$\begin{aligned} \mathbf{j} &= \frac{\hbar}{m} \text{Im}(\Psi^* \nabla \Psi) \\ &= \frac{\hbar}{m} \text{Im} \left\{ (x - iy) e^{-A(x^2+y^2)/2} e^{i\omega t} \left( \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} \right) \left[ (x + iy) e^{-A(x^2+y^2)/2} e^{-i\omega t} \right] \right\} \\ &= \frac{\hbar}{m} \text{Im} \left\{ (x - iy) e^{-A(x^2+y^2)/2} \left( \hat{\mathbf{i}} [1 - Ax(x + iy)] + \hat{\mathbf{j}} [i - Ay(x + iy)] \right) e^{-A(x^2+y^2)/2} \right\} \\ &= \frac{\hbar}{m} e^{-A(x^2+y^2)} \text{Im} \left\{ \hat{\mathbf{i}} [x - iy - Ax(x^2 + y^2)] + \hat{\mathbf{j}} [ix + y - Ay(x^2 + y^2)] \right\} \\ &= \frac{\hbar}{m} (-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}) e^{-A(x^2+y^2)}. \end{aligned}$$

- (c) [3] Check the local version of conservation of probability, *i.e.*, show that your solution satisfies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Since  $\rho$  is independent of time, the first term is zero.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} &= \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = \frac{\hbar}{m} \left\{ \frac{\partial}{\partial x} \left[ -ye^{-A(x^2+y^2)} \right] + \frac{\partial}{\partial y} \left[ xe^{-A(x^2+y^2)} \right] \right\} \\ &= \frac{\hbar}{m} \left[ 2Axye^{-A(x^2+y^2)} - 2Axye^{-A(x^2+y^2)} \right] = 0 \end{aligned}$$

2. [10] For this problem, I want you to prove Schwartz's inequality,

$$(\phi, \psi)(\psi, \phi) \leq (\phi, \phi)(\psi, \psi)$$

**You may prove it however you want; however, here is one way to prove it. Expand out the inner product of  $a\phi + b\psi$  with itself, which must be positive, where  $a$  and  $b$  are arbitrary *complex* numbers. Then substitute in  $a = (\phi, \psi)$  and  $b = -(\phi, \phi)$ . Simplify, and you should have the desired result.**

We take the suggestion given, hoping it will not lead us astray. We note that  $b$  is real, so  $b^* = b = -(\phi, \phi)$ , while  $a$  is not, so  $a^* = (\phi, \psi)^* = (\psi, \phi)$

$$\begin{aligned} 0 &\leq (a\phi + b\psi, a\phi + b\psi) \\ &= a^*a(\phi, \phi) + a^*b(\phi, \psi) + b^*a(\psi, \phi) + b^*b(\psi, \psi) \\ &= (\psi, \phi)(\phi, \psi)(\phi, \phi) - (\psi, \phi)(\phi, \phi)(\phi, \psi) - (\phi, \phi)(\phi, \psi)(\psi, \phi) + (\phi, \phi)(\phi, \phi)(\psi, \psi) \\ &= -(\phi, \phi)(\phi, \psi)(\psi, \phi) + (\phi, \phi)(\phi, \phi)(\psi, \psi) \end{aligned}$$

We now rearrange this to give

$$\begin{aligned} (\phi, \phi)(\phi, \psi)(\psi, \phi) &\leq (\phi, \phi)(\phi, \phi)(\psi, \psi) \\ (\phi, \psi)(\psi, \phi) &\leq (\phi, \phi)(\psi, \psi) \end{aligned}$$

The last line is the desired relationship. The only detail that might be unclear is that in the ultimate step, we divided by  $(\phi, \phi)$ . This is valid, provided  $(\phi, \phi) > 0$ , which is guaranteed for  $\phi \neq 0$ . Of course, if  $\phi = 0$ , then both sides of Schwartz's inequality are zero, and the result is trivially true.

3. [15] Consider the set of all functions that can be expressed as a power series on the region  $[-1,1]$ , that is, the set of all functions of the form

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \quad \text{for } -1 < x < 1$$

The inner product of two such functions would then be defined as

$$(f, g) = \int_{-1}^1 f^*(x) g(x) dx$$

It is clear that the set of functions  $\{1, x, x^2, \dots\}$  form a basis for these functions.

However, it is not an orthonormal basis. Let's use our ket notation and label these kets  $\{|0\rangle, |1\rangle, |2\rangle, \dots\}$ , so that  $|n\rangle$  corresponds to the function  $x^n$ , or another way to put it,  $\langle x|n\rangle = x^n$ .

- (a) [3] Find the inner product  $\langle n|m\rangle$  for any two of these states.

We have

$$\begin{aligned} \langle n|m\rangle &= \int_{-1}^1 x^n x^m dx = \frac{1}{n+m+1} x^{n+m+1} \Big|_{-1}^1 = \frac{1}{n+m+1} [1 - (-1)^{n+m+1}] \\ &= \begin{cases} 2/(n+m+1) & \text{if } n+m \text{ even} \\ 0 & \text{if } n+m \text{ odd} \end{cases} \end{aligned}$$

- (b) [4] Using the procedure discussed in class, define an orthogonal basis  $\{|n'\rangle\}$ .

Work out  $|n'\rangle$  explicitly for  $n = 0, 1, 2$ , and  $3$ .

We will simply use the procedure laid out in class

$$|0'\rangle = |0\rangle$$

$$|1'\rangle = |1\rangle - \frac{\langle 0'|1\rangle}{\langle 0'|0'\rangle} |0'\rangle = |1\rangle - \frac{\langle 0|1\rangle}{\langle 0|0\rangle} |0\rangle = |1\rangle - 0 = |1\rangle$$

$$|2'\rangle = |2\rangle - \frac{\langle 0'|2\rangle}{\langle 0'|0'\rangle} |0'\rangle - \frac{\langle 1'|2\rangle}{\langle 1'|1'\rangle} |1'\rangle = |2\rangle - \frac{\langle 0|2\rangle}{\langle 0|0\rangle} |0\rangle - \frac{\langle 1|2\rangle}{\langle 1|1\rangle} |1\rangle = |2\rangle - \frac{2/3}{2} |0\rangle - 0 = |2\rangle - \frac{1}{3} |0\rangle$$

$$\begin{aligned} |3'\rangle &= |3\rangle - \frac{\langle 0'|3\rangle}{\langle 0'|0'\rangle} |0'\rangle - \frac{\langle 1'|3\rangle}{\langle 1'|1'\rangle} |1'\rangle - \frac{\langle 2'|3\rangle}{\langle 2'|2'\rangle} |2'\rangle = |3\rangle - \frac{\langle 0|3\rangle}{\langle 0|0\rangle} |0\rangle - \frac{\langle 1|3\rangle}{\langle 1|1\rangle} |1\rangle - \frac{\langle 2|3\rangle - \frac{1}{3}\langle 0|3\rangle}{\langle 2'|2'\rangle} |2'\rangle \\ &= |3\rangle - 0 - \frac{2/5}{2/3} |1\rangle - 0 = |3\rangle - \frac{3}{5} |1\rangle \end{aligned}$$

(c) [5] Define the orthonormal basis  $\{|n''\rangle\}$  as in class, and work out  $|n''\rangle$  for  $n = 0, 1, 2$ , and 3.

We simply follow the procedure as in class again, which gives us

$$|0''\rangle = \frac{|0'\rangle}{\sqrt{\langle 0'|0'\rangle}} = \frac{|0\rangle}{\sqrt{\langle 0|0\rangle}} = \frac{|0\rangle}{\sqrt{2}} = \sqrt{\frac{1}{2}}|0\rangle$$

$$|1''\rangle = \frac{|1'\rangle}{\sqrt{\langle 1'|1'\rangle}} = \frac{|1\rangle}{\sqrt{\langle 1|1\rangle}} = \frac{|1\rangle}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}|1\rangle$$

$$\begin{aligned} |2''\rangle &= \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{|2\rangle - \frac{1}{3}|0\rangle}{\sqrt{(\langle 2| - \frac{1}{3}\langle 0|)(|2\rangle - \frac{1}{3}|0\rangle)}} = \frac{|2\rangle - \frac{1}{3}|0\rangle}{\sqrt{\langle 2|2\rangle - \frac{1}{3}\langle 2|0\rangle - \frac{1}{3}\langle 0|2\rangle + \frac{1}{9}\langle 0|0\rangle}} \\ &= \frac{|2\rangle - \frac{1}{3}|0\rangle}{\sqrt{\frac{2}{5} - \frac{1}{3} \cdot \frac{2}{3} - \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{9} \cdot 2}} = \sqrt{\frac{45}{8}}(|2\rangle - \frac{1}{3}|0\rangle) = \sqrt{\frac{5}{2}}(\frac{3}{2}|2\rangle - \frac{1}{2}|0\rangle) \end{aligned}$$

$$\begin{aligned} |3''\rangle &= \frac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}} = \frac{|3\rangle - \frac{3}{5}|1\rangle}{\sqrt{(\langle 3| - \frac{3}{5}\langle 1|)(|3\rangle - \frac{3}{5}|1\rangle)}} = \frac{|3\rangle - \frac{3}{5}|1\rangle}{\sqrt{\langle 3|3\rangle - \frac{3}{5}\langle 3|1\rangle - \frac{3}{5}\langle 1|3\rangle + \frac{9}{25}\langle 1|1\rangle}} \\ &= \frac{|3\rangle - \frac{3}{5}|1\rangle}{\sqrt{\frac{7}{2} - \frac{3}{5} \cdot \frac{2}{5} - \frac{3}{5} \cdot \frac{2}{5} + \frac{9}{25} \cdot \frac{2}{3}}} = \sqrt{\frac{175}{8}}(|3\rangle - \frac{3}{5}|1\rangle) = \sqrt{\frac{7}{2}}(\frac{5}{2}|3\rangle - \frac{3}{2}|1\rangle) \end{aligned}$$

(d) [2] Write out the expressions  $\langle x|n''\rangle$  explicitly for  $n = 0, 1, 2, 3$  as polynomials, and compare them to the Legendre polynomials  $P_n(x)$ . If you have made no mistake, the relationship should be very simple.

You can find out about Legendre polynomials from any number of sources; for example, if you Google “Legendre” the first website you will find will be

<http://mathworld.wolfram.com/LegendrePolynomial.html>

This gives explicit forms for them. We then have

$$\langle x|0''\rangle = \sqrt{\frac{1}{2}}\langle x|0\rangle = \sqrt{\frac{1}{2}}P_0(x)$$

$$\langle x|1''\rangle = \sqrt{\frac{3}{2}}\langle x|1\rangle = \sqrt{\frac{3}{2}}x = \sqrt{\frac{3}{2}}P_1(x)$$

$$\langle x|2''\rangle = \sqrt{\frac{5}{2}}(\frac{3}{2}\langle x|2\rangle - \frac{1}{2}\langle x|0\rangle) = \sqrt{\frac{5}{2}}(\frac{3}{2}x^2 - \frac{1}{2}) = \sqrt{\frac{5}{2}}P_2(x)$$

$$\langle x|3''\rangle = \sqrt{\frac{7}{2}}(\frac{5}{2}\langle x|3\rangle - \frac{3}{2}\langle x|1\rangle) = \sqrt{\frac{7}{2}}(\frac{5}{2}x^3 - \frac{3}{2}x) = \sqrt{\frac{7}{2}}P_3(x)$$

It is not difficult to see that the pattern is  $\langle x|n''\rangle = \sqrt{\frac{2n+1}{2}}P_n(x)$ .