Physics 741 – Graduate Quantum Mechanics 1

Solution Set A

1. **[10] A particle of mass *m* lies in one-dimension in a potential of the form *V*(*x*) = *Fx,* where *F* is constant. The wave function at time *t* is given by**

$$\Psi(x,t) = N(t)\exp\left[-\tfrac{1}{2}A(t)x^2 + B(t)x\right]$$

where *N, A,* and *B* are all *complex* functions of time. Use Schrödinger's equation to derive equations for the time derivative of the three functions *A, B,* and *N.* You do not need to solve these equations.

We first work out the time derivative and two space derivatives.

$$\frac{\partial}{\partial t}\psi(x) = \left(\dot{N} - \tfrac{1}{2}N\dot{A}x^2 + N\dot{B}x\right)\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right),$$

$$\frac{\partial}{\partial x}\psi(x) = N\left(-Ax + B\right)\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right),$$

$$\frac{\partial^2}{\partial x^2}\psi(x) = N\left[-A + \left(-Ax + B\right)^2\right]\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right),$$

Now we simply plug these results into Schrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x)\psi,$$

$$i\hbar\left(\dot{N} - \tfrac{1}{2}N\dot{A}x^2 + N\dot{B}x\right)\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right) = -\frac{\hbar^2}{2m}N\left[-A + \left(-Ax + B\right)^2\right]\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right)$$
$$+FxN\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right).$$

Canceling the common exponential and dividing by $i\hbar N$, this simplifies to

$$\frac{\dot{N}}{N} - \tfrac{1}{2}\dot{A}x^2 + \dot{B}x = \frac{i\hbar}{2m}\left(A^2x^2 - 2ABx + B^2 - A\right) - \frac{iF}{\hbar}x.$$

This expression must be true at all positions x. The only way this can happen is if the coefficients of x^2, x, and the constant terms all match on the two sides of the equation. This implies

$$\dot{A} = -\frac{i\hbar}{m}A^2, \qquad \dot{B} = -\frac{i\hbar}{m}AB - \frac{iF}{\hbar}, \qquad \dot{N} = iN\frac{\hbar}{2m}\left(B^2 - A\right).$$

The first of these is, in fact, pretty easy to solve, but the others are a bit trickier.

$$\frac{1}{A(t)} = \frac{1}{A_0} + \frac{i\hbar}{m}t.$$

2. **[10] For each of the wave functions in one dimension given below, *N* and *a* are positive real numbers. Determine the normalization constant *N* (in terms of *a*), and determine the probability that a measurement of the position of the particle will yield a result *x* > *a*.**

For each of these parts, we will use the normalization condition $\int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx = 1$ to get the normalization constraint, then we'll simply change the lower limit of integration to a to get the probability. If any integrals get tricky, we can ask Maple to do them for us.

(a) [4] $\psi(x) = \dfrac{N}{x+ia}$

$$1 = \int_{-\infty}^{\infty}\left|\psi(x)\right|^2 dx = \int_{-\infty}^{\infty}\psi^*(x)\psi(x)\,dx = N^2\int_{-\infty}^{\infty}\frac{dx}{(x+ia)(x-ia)} = N^2\int_{-\infty}^{\infty}\frac{dx}{x^2+a^2}$$

$$= \left(N^2/a\right)\tan^{-1}(x/a)\Big|_{-\infty}^{\infty} = \pi N^2/a,$$

$$N = \sqrt{a/\pi}$$

$$P(x>a) = \left(N^2/a\right)\tan^{-1}(x/a)\Big|_{a}^{\infty} = \frac{\tfrac{1}{2}\pi - \tfrac{1}{4}\pi}{\pi} = \tfrac{1}{4} = 25\%$$

(b) [3] $\psi(x) = N\exp\left(-|x|/a\right)$

$$1 = N^2\int_{-\infty}^{\infty} e^{-2|x|/a}dx = 2N^2\int_0^{\infty} e^{-2x/a}dx = N^2 a\, e^{-2x/a}\Big|_0^{\infty} = N^2 a,$$

$$N = 1/\sqrt{a}$$

$$P(x>a) = N^2\int_a^{\infty} e^{-2|x|/a}dx = \tfrac{1}{2}N^2 a\, e^{-2x/a}\Big|_a^{\infty} = \tfrac{1}{2}e^{-2} = 6.767\%$$

(c) [3] $\psi(x) = \begin{cases} Nx^2(x-2a) & \text{for } 0<x<2a \\ 0 & \text{otherwise}\end{cases}$

$$1 = N^2\int_0^{2a}\left[x^2(x-2a)\right]^2 dx = N^2\int_0^{2a}\left(x^6 - 4ax^5 + 4a^2x^4\right)dx$$

$$= N^2\left(\tfrac{1}{7}x^7 - \tfrac{2}{3}ax^6 + \tfrac{4}{5}a^2x^5\right)\Big|_0^{2a} = \tfrac{128}{105}N^2a^7$$

$$N = \sqrt{\tfrac{105}{128}}a^{-7/2}$$

$$P(x>a) = N^2\int_a^{2a}\left[x^2(x-2a)\right]^2 dx = N^2\left(\tfrac{1}{7}x^7 - \tfrac{2}{3}ax^6 + \tfrac{4}{5}a^2x^5\right)\Big|_a^{2a}$$

$$= \tfrac{105}{128}a^{-7}\left(\tfrac{128}{105}a^7 - \tfrac{29}{105}a^7\right) = \tfrac{99}{128} = 77.34\%$$

3. [5] An electron in the ground state of hydrogen has, in spherical coordinates, the wave function

$$\psi(r,\theta,\phi) = Ne^{-r/a}$$

where *N* and *a* are positive constants. Determine the normalization constant *N* and the probability that a measurement of the position will yield *r* > *a*. Don't forget you are working in three dimensions!

In three dimensions, when working in spherical coordinates, the normalization condition is

$$1 = \iiint \left|\psi(\mathbf{r})\right|^2 d^3\mathbf{r} = \int_0^\infty r^2 dr \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \left|\psi(\mathbf{r})\right|^2$$

In this case, the wave function depends only on *r*, so the inner two integrals are trivial.

$$1 = 4\pi N^2 \int_0^\infty r^2 e^{-2r/a} dr = 4\pi N^2 \left(-\tfrac{1}{2}r^2 a - \tfrac{1}{2}ra^2 - \tfrac{1}{4}a^3\right) e^{-2r/a} \Big|_0^\infty = \pi a^3 N^2$$

$$N = 1\Big/\sqrt{\pi a^3}$$

$$P(r > a) = 4\pi N^2 \int_a^\infty r^2 e^{-2r/a} dr = \frac{4}{a^3}\left(-\tfrac{1}{2}r^2 a - \tfrac{1}{2}ra^2 - \tfrac{1}{4}a^3\right) e^{-2r/a} \Big|_a^\infty = 5e^{-2} = 67.67\%$$