Physics 741 – Graduate Quantum Mechanics 1

Homework Set V

Due Wednesday, November 5

1. This problem has nothing to do with quantum mechanics. In the presence of a charged plasma, it is possible to create electromagnetic waves that are longitudinal, having electric polarization parallel to the direction of propagation. The fields take the form

$$\mathbf{E}(\mathbf{r},t) = E_0\hat{\mathbf{k}}\cos(kz-\omega t), \qquad \mathbf{B}(\mathbf{r},t) = 0 .$$

   (a) Show that this electric field (and lack of magnetic field) can be written purely in terms of a vector potential, without the use of the scalar potential; that

$$\mathbf{A}_1(\mathbf{r},t) = \hat{\mathbf{k}}A_1(\mathbf{r},t) \quad \text{and} \quad U_1(\mathbf{r},t) = 0$$

   and determine the function $A_1(\mathbf{r},t)$ that makes this work.

   (b) Show that the same electric fields (and lack of magnetic field) can also be written purely in terms of a scalar potential,

$$\mathbf{A}_2(\mathbf{r},t) = 0 \quad \text{and} \quad U_2(\mathbf{r},t)$$

   and determine the scalar potential $U_2(\mathbf{r},t)$ that makes this work.

   (c) Show that these two sets of potential, $(\mathbf{A}_1, U_1)$ and $(\mathbf{A}_2, U_2)$, are related by a gauge transformation, and determine explicitly the form of the gauge function $\chi(\mathbf{r},t)$ that relates them.

2. In chapter three, we defined the probability density ρ and probability current $\mathbf{j}$ as

$$\rho = \Psi^*\Psi \quad \text{and} \quad \mathbf{j} = \hbar\left(-i\Psi^*\nabla\Psi + i\Psi\nabla\Psi^*\right)/2m = \left(\Psi^*\mathbf{P}\Psi - \Psi\mathbf{P}\Psi^*\right)/2m$$

   and then derived the conservation of probability formula

$$\partial\rho/\partial t + \nabla\cdot\mathbf{j} = 0$$

   from Schrödinger's equation (3.1b). However, Schrödinger's equation has just changed into (10.23), and our proof is no longer valid.

   (a) Which of the quantities ρ and $\mathbf{j}$ is invariant under a gauge transformation?

   (b) Show that the derivation of the conservation law is no longer valid.

   (c) Redefine the probability current $\mathbf{j}$ by replacing $\mathbf{P} \to \boldsymbol{\pi}$, defined by

$$\boldsymbol{\pi}\Psi = (\mathbf{P} + e\mathbf{A})\Psi \quad \text{and} \quad \boldsymbol{\pi}\Psi^* = (\mathbf{P} - e\mathbf{A})\Psi^*$$

   Show that the new probability current $\mathbf{j}$ *is* gauge invariant.

   (d) Show that with this definition of the current $\mathbf{j}$, conservation of probability works.