Physics 741 – Graduate Quantum Mechanics 1

Homework Set R

Due Friday, October 24

1. Suppose we have a particle with spin ½. Let's define the spin operator S_θ as

$$S_\theta = \tfrac{1}{2}\hbar\left(\cos\theta\sigma_z + \sin\theta\sigma_x\right)$$

In other words, we are measuring the spin of the particle along an axis that is at an angle θ compared to the *z*-axis

(a) Verify that

$$\left|+_\theta\right\rangle = \cos\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \sin\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle$$
$$\left|-_\theta\right\rangle = -\sin\left(\tfrac{1}{2}\theta\right)\left|+_z\right\rangle + \cos\left(\tfrac{1}{2}\theta\right)\left|-_z\right\rangle$$

are normalized eigenvectors of S_θ, and determine the eigenvalues. In other words, you have to demonstrate 3 things, (i) they are normalized, (ii) they are eigenvectors, (iii) determine their eigenvalues.

(b) Suppose a particle is initially in the state $\left|+_z\right\rangle$. If a subsequent measurement at angle θ is done, what is the probability the result will come out positive? What is the state immediately after the measurement is done?

(c) After the measurement at angle θ yields a positive result, *another* measurement is done, this time at angle θ' is performed. What is the probability this time that the result comes out positive?

2. It is common to have to calculate matrix elements of the form

$$\left\langle n', l', m_l', m_s'\right|\mathbf{L}\cdot\mathbf{S}\left|n, l, m_l, m_s\right\rangle,$$

where **L** and **S** are the orbital and spin angular momenta respectively, and l, m_l, and m_s are quantum numbers corresponding to the operators $\mathbf{L}^2$, L_z, and S_z, respectively (n represents some sort of radial quantum number).

(a) Show that $\mathbf{L}\cdot\mathbf{S}$ can be written in a simple way in terms of $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2 = \left(\mathbf{L}+\mathbf{S}\right)^2$. You may assume any commutation relations that you know are true about **L** and**S**, or that you proved in a previous problem set.

(b) Show that the operator $\mathbf{L}\cdot\mathbf{S}$ commutes with $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$.

(c) A more intelligent basis to use would be eigenstates of $\mathbf{L}^2$, $\mathbf{S}^2$, and $\mathbf{J}^2$, and J_z, so our states would look like $\left|n, l, j, m_j\right\rangle$ (the constant *s* is implied). Assuming our states are orthonormal, work out a simple formula for

$$\left\langle n', l', j', m_j'\right|\mathbf{L}\cdot\mathbf{S}\left|n, l, j, m_j\right\rangle$$

(d) For arbitrary l =0, 1, 2, … and s = ½, what are the possible values of j? Work out all matrix elements of the form above in this case.