Physics 741 – Graduate Quantum Mechanics 1

Homework Set P
Due Monday, October 20

1. In class we solved the hydrogen atom, which is a spherical $1/r$ potential. Consider another spherically symmetric potential, namely, the spherical harmonic oscillator

$$H = \frac{\mathbf{P}^2}{2m} + \frac{1}{2} m\omega^2 \mathbf{Q}^2$$

This potential is most easily solved by separation of variables, but it is very helpful to take advantage of the spherical symmetry to find solutions.

(a) Factor eigenstates of this Hamiltonian into the form $\psi(r,\theta,\phi) = R(r) Y_l^m(\theta,\phi)$. Find a differential equation satisfied by the radial wave function $R(r)$.

(b) At large r, which term besides the derivative term dominates? Show that for large r, we can satisfy the differential equation if $R(r) \sim \exp\left(-\tfrac{1}{2} A r^2\right)$, and determine the factor A that will make this work.

(c) Write the radial wave function in the form $R(r) = f(r)\exp\left(-\tfrac{1}{2} A r^2\right)$, and show that f must satisfy

$$\frac{2mE}{\hbar^2} f = -\frac{1}{r}\frac{d^2}{dr^2}(fr) + 2A\frac{d}{dr}(fr) + Af + \frac{l^2+l}{r^2} f .$$

(d) Assume that at small r, the wave function goes like $f(r) \sim r^k$. What value of k will make this equation work?

(e) Assume that the radial wave function can be written as a power series, similar to what we did in class,

$$f(r) = \sum_{i=k}^{n} f_i r^i .$$

Substitute this into the differential equation for f, and thereby discover a recursion relation on the f_i's. Unlike the recursion relationship we found, you will get a recursion relationship relating f_i to f_{i+2}. Hence the series actually requires only odd power of r or even powers of r, not both.

(f) Assume, as in class, that the series terminates, so that f_n is the last term, and hence that f_{n+2} vanishes. Find a condition for the energy E in terms of n.

(g) Given n, which describes the energy of the atom, what restrictions are there on l, the total angular momentum quantum number?