Physics 741 – Graduate Quantum Mechanics 1

Homework Set N

Due Monday, October 13

1. Let $\mathbf{J} = \left(J_x, J_y, J_z\right)$ be three Hermitian operators that commute with the Hamiltonian H, and have angular momentum-like commutation relations

$$\left[J_x, J_y\right] = i\hbar J_z, \qquad \left[J_y, J_z\right] = i\hbar J_x, \qquad \left[J_z, J_x\right] = i\hbar J_y.$$

   Define the operators

$$\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2 \qquad \text{and} \qquad J_\pm = J_x \pm iJ_y$$

   Show each of the following is true:

   (a) $\left[\mathbf{J}^2, \mathbf{J}\right] = 0$ (this is three identities), and therefore $\left[J^2, J_\pm\right] = 0$

   (b) $\left[J_z, J_\pm\right] = \pm\hbar J_\pm$

   (c) $\mathbf{J}^2 = J_\mp J_\pm + J_z^2 \pm \hbar J_z$

2. For $j = 2$, we will work out the explicit form for all of the matrices $\mathbf{J}$.

   (a) Write out the expression for J_z and $J_\pm$ as an appropriately sized matrix.

   (b) Write out J_x and J_y.

   (c) Check explicitly that $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ is a constant matrix with the appropriate value.

3. The Pauli matrices are defined in equation (8.30). Using the Pauli matrices, show that

   (a) $\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right)^2 = 1$ for any unit vector $\hat{\mathbf{r}}$

   (b) $\exp\left(-\tfrac{1}{2}i\theta\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right) = \cos\left(\tfrac{1}{2}\theta\right) - i\sin\left(\tfrac{1}{2}\theta\right)\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right)$