Physics 741 – Graduate Quantum Mechanics 1

Homework Set H

Due Friday, September 19

1. It is sometimes said that a watched kettle never boils. In some sense, this is true in quantum mechanics. Consider a quantum system where the state space is two dimensional, with basis states $\{|0\rangle, |1\rangle\}$, the former representing the kettle in the "not boiled" state, the latter the "boiled" state. In terms of these, the Hamiltonian is given by

$$H = \hbar\omega \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

   (a) At *t*, the quantum state is given by

$$|\Psi(t=0)\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   Solve Schrödinger's equation given the initial conditions, and determine the ket of the kettle at later times,

$$|\Psi(t)\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

   (b) At time $t = \Delta t$, an observer checks whether the kettle has boiled yet. That is, he measures the quantum system using the boiled operator *B*, defined by $B|0\rangle = 0|0\rangle$ and $B|1\rangle = 1|1\rangle$. What is the probability P_1 that the kettle has boiled at this time (*i.e.*, that measuring *B* yields the eigenvalue 1)? If the kettle is boiled at this time, the total time $T = \Delta t$ is recorded.

   (c) If the kettle is *not* boiled, what is the quantum state immediately *after* the measurement has been made?

   (d) After a second interval of Δt, the kettle is measured again to see if it has boiled. What is the probability P_2 that it is not boiled the first time, and it is boiled the second? If this occurs, the total time $T = 2\Delta t$ is recorded.

   (e) The process is repeated until the kettle has actually boiled. What is the general formula P_n that it first boils on the *n*'th measurement? Write a formula for the average time $\langle T \rangle = \langle n\Delta t \rangle$ that it takes for the kettle to boil. The formula below may be helpful

   (f) Demonstrate that in the limit $\Delta t \to 0$, it takes forever for the kettle to boil.

   (g) Determine (numerically or otherwise) the optimal time Δt so that $\langle T \rangle = \langle n\Delta t \rangle$ will be minimized, so the kettle boils as quickly as possible.

Helpful formula: $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) = \frac{1}{(1-x)^2}$