Physics 741 – Graduate Quantum Mechanics 1

Homework Set β
Due Monday, November 24

1. At $t = 0$, the wave function of a free particle is given by

$$\Psi(x,t=0) = (A/\pi)^{1/4} \exp\left[-\tfrac{1}{2}A(x-x_0)^2\right]$$

Find $\Psi(x,t)$, using the propagator.

2. This problem should be worked entirely in the Heisenberg formulation of quantum mechanics. A particle lies in the one-dimensional harmonic oscillator potential:

$$H = \frac{P^2}{2m} + \tfrac{1}{2}m\omega^2 Q^2$$

(a) Work out dQ/dt and dP/dt.

(b) Define the operators

$$a(t) = \sqrt{\frac{m\omega}{2\hbar}}Q(t) + i\sqrt{\frac{1}{2\hbar m\omega}}P(t) \qquad \text{and} \qquad a^\dagger(t) = \sqrt{\frac{m\omega}{2\hbar}}Q(t) - i\sqrt{\frac{1}{2\hbar m\omega}}P(t)$$

Show that these satisfy equations $\dot{a}(t) \propto a(t)$ and $\dot{a}^\dagger(t) \propto a^\dagger(t)$.

(c) Solve the differential equations for $a(t)$ and $a^\dagger(t)$ in terms of $a(0)$ and $a^\dagger(0)$. As a check, confirm that the Hamiltonian is independent of time:

$$H = \hbar\omega\left[a^\dagger(t)a(t) + \tfrac{1}{2}\right],$$

(d) Rewrite $Q(t)$ and $P(t)$ in terms of $a(t)$ and $a^\dagger(t)$, and rewrite $a(0)$ and $a^\dagger(0)$ in terms of $Q(0)$ and $P(0)$, so that $Q(t)$ and $P(t)$ depend only on $X(0)$ and $P(0)$. You may find the identities below useful.

$$Q(t) = \sqrt{\frac{\hbar}{2m\omega}}\left[a(t) + a^\dagger(t)\right] \quad \text{and} \quad P(t) = i\sqrt{\frac{\hbar m\omega}{2}}\left[a^\dagger(t) - a(t)\right].$$

As a check, you should find $Q(T) = Q(0)$, if T is the classical period.

(e) Suppose the quantum state (which is independent of time) is chosen to be an eigenstate of $Q(0)$,

$$Q(0)|\psi\rangle = x_0|\psi\rangle.$$

Show that at the times $t = \frac{1}{4}T$, $t = \frac{1}{2}T$, $t = \frac{3}{4}T$, and $t = T$, it is an eigenstate of either $Q(t)$ *or* $P(t)$, and determine its eigenvalue.