

Final Exam, Spring 2010

Due Friday, May 7

You may use (1) class notes, (2) the text, (3) former homeworks and solutions (available online), or (4) any math references, such as integral tables, Maple, etc, including any routines I provided for you (5) any source for relevant constants. If you cannot do an integral, or if you cannot solve an equation, try to go on, as if you knew the answer. Some useful integral appear on the last page. Feel free to contact me with questions.

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1. **[30]** An experimenter is attempting to place a particle in a superposition state of a harmonic oscillator with angular frequency ω at time t . It will be in some combination of the $|0\rangle$ and $|1\rangle$ state, though we are unsure which combination, and therefore we will use a state operator ρ to describe it, which we represent as a 2×2 matrix:

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the first row/column corresponds to the $|0\rangle$ state and the second to the $|1\rangle$ state.

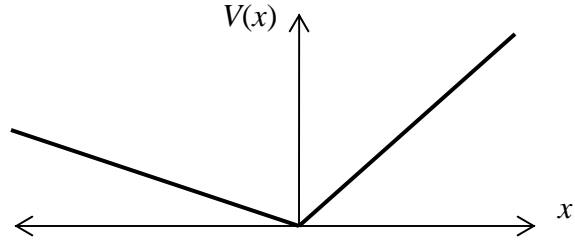
- (a) **[4]** Given the known properties of the state matrix, show that two of these constants are real and two are complex conjugates of each other. Also find a relation between the two real parts.
 - (b) **[8]** What is the Hamiltonian represented as a matrix in this basis? Write explicitly an expression for $d\rho/dt$ using Schrödinger's Equation in this basis.
 - (c) **[5]** Show that two of the components are independent of time. Show that the other two will have solution $g = g_0 e^{\pm i\omega t}$, where g represents some of these parameters and g_0 is its value at $t = 0$.
 - (d) **[8]** At $t = 0$, the harmonic oscillator is placed in the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/3}|1\rangle)$ or $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\pi/3}|1\rangle)$ with equal probability. What is the state matrix ρ at $t = 0$?
 - (e) **[5]** What is the state matrix at all times $\rho(t)$?
2. **[15]** A particle of mass m is in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$. Estimate the ground state energy using the variational principle with the trial wave function

$$\psi(x) = \begin{cases} a^2 - x^2 & \text{for } -a < x < a, \\ 0 & \text{otherwise} \end{cases}$$

where a is a variational parameter. Find the ratio of this estimate to the actual ground state energy. As a check, this ratio should be a pure number.

3. [10] A particle of mass m in one dimension lies in the asymmetric potential

$$V(x) = \begin{cases} -\alpha x & \text{if } x < 0, \\ \beta x & \text{if } x > 0. \end{cases}$$



as sketched at right.

- (a) [3] If the particle has energy E , find the classical turning points a and b .
- (b) [7] Using the WKB approximation, find estimates of the energy E_n of the n 'th eigenstate.
4. [35] A particle of mass m lies in a three dimensional cubical infinite square well, with potential

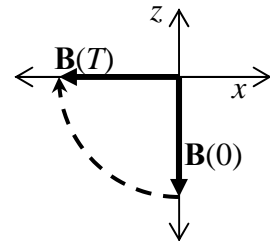
$$V(x, y, z) = \begin{cases} 0 & \text{if } 0 < x, y, z < a \\ \infty & \text{otherwise} \end{cases}$$

To this is added a small perturbation

$$W(x, y, z) = 4\Delta \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right)$$

- (a) [5] Find the exact eigenstates and eigenenergies of the unperturbed Hamiltonian. Write explicitly the properly normalized wave functions $\psi_{ijk}(x, y, z)$.
- (b) [15] Find the energy of the ground state to second order in the perturbation Δ . Some possibly useful integrals can be found at the end of this test. Also, write the ground state wave function to first order in the perturbation Δ .
- (c) [15] For the first excited state(s), find the eigenstates of the full Hamiltonian to leading order, and the energies of these states to first order in Δ .
5. [10] An electron lies in a strong magnetic field $\mathbf{B}(t)$ whose magnitude is constant, but whose direction changes. The Hamiltonian (considering only spin) is therefore

$$H = \frac{e}{m} \mathbf{B}(t) \cdot \mathbf{S}$$



At $t = 0$, the magnetic field is pointing in the $-\hat{z}$ direction ($\mathbf{B}(t=0) = -B_0\hat{z}$), and the electron is in the ground state $|+_z\rangle$, an eigenstate of S_z , with eigenvalue $+\frac{1}{2}\hbar$.

- (a) [4] The direction of the magnetic field is rotated slowly and uniformly, until at $t = T$ it is pointing in the $-\hat{x}$ direction ($\mathbf{B}(t=T) = -B_0\hat{x}$). What is the probability that it is still in the ground state?
- (b) [6] The magnetic field is now *suddenly* changed to point once again in the $-\hat{z}$ direction. What is the probability *now* that it is still in the ground state?

6. **[30]** A particle of mass m and energy $E = \hbar^2 k^2 / 2m$ scatters from a spherically symmetric potential $V(\mathbf{r}) = \alpha / r^2$, where α is a positive constant. We will calculate the cross section using the method of partial waves.

- (a) **[3]** Find a differential equation satisfied by the radial wave function $R_l(r)$, the radial wave functions.
- (b) **[10]** Show that this equation is satisfied by the function $R_l(r) = a j_\nu(kr) + b n_\nu(kr)$ where $j_\nu(x)$ and $n_\nu(x)$ are spherical Bessel functions, and a and b are arbitrary constants. You will find a quadratic equation for ν which you must solve (choose the positive solution only). Normally we work with integer-indexed spherical Bessel functions, but there is nothing wrong with using non-integer-indexed spherical Bessel functions.
- (c) **[5]** Assume that $j_\nu(x)$ and $n_\nu(x)$ have similar asymptotic properties to $j_l(x)$ and $n_l(x)$, *i.e.*, if one of them diverges at the origin, the other will, and formulas that describe their large x behavior still work. If our wave function does not diverge at the origin, what does this tell us about a and/or b ? At large r , what is the behavior of $R_l(r)$?
- (d) **[5]** The radial wave function must behave like $R_l(r) \propto \sin(kr - \frac{1}{2}\pi l + \delta_l) / kr$ at large r . Using this, show that

$$\delta_l = \frac{1}{2}\pi \left[l + \frac{1}{2} - \sqrt{l^2 + l + \frac{1}{4} + 2m\alpha/\hbar^2} \right]$$

- (e) **[7]** Find the phase shift for the first two modes, δ_0 and δ_1 , if $m\alpha/\hbar^2 = 3$. Using these, estimate the differential cross-section $d\sigma/d\Omega$. (Comment: this is actually inaccurate, because large l contributions are important, but do it anyway).

7. **[20]** An electron is in the ground state $|100\rangle$ of hydrogen at $t = -\infty$. In an attempt to excite it into one of the excited states $|21m\rangle$, it is hit with a short electromagnetic pulse which adds a potential $W(t) = FZe^{-\lambda|t|}$, where F is a small constant.

- (a) **[7]** Calculate the matrix elements $\langle 21m | W(t) | 100 \rangle$, and argue that only one value of m will become populated by this process.
- (b) **[13]** Calculate the probability $P(100 \rightarrow 21m)$ in the limit $t \rightarrow \infty$ to leading order in the perturbation F . You may write your answers in terms of ω_{F1} , if you wish, but make it clear what ω_{F1} is.

8. [30] A state vector contains only one type of photon, but the number of photons is uncertain.

$$|\psi\rangle = N \sum_{n=0}^{\infty} 2^{-n} |n, \mathbf{q}, \tau\rangle$$

There are a couple of possibly useful formulas at the end of this problem.

- (a) [5] What is the correct normalization N ?
- (b) [10] What is the expectation value of the energy for this state, $\langle H \rangle$? Use the convention that the energy of the vacuum is zero.
- (c) [15] What is the expectation value of the electric field, $\langle \mathbf{E}(\mathbf{r}) \rangle$? To simplify, you can assume $\epsilon_{\mathbf{q}\tau}$ is real. You may leave one infinite sum undone, but this sum should be a pure number.

Possibly helpful formulas: If $|x| < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad \sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^2}.$$

9. [20] An electron is trapped in a 3D infinite square well with potential

$$V(x, y, z) = \begin{cases} 0 & \text{if } 0 < x, y, z < a \\ \infty & \text{otherwise} \end{cases}$$

You should already have the energy eigenvalues and wave functions from problem 4. It is initially in the state $|113\rangle$.

- (a) [10] Show that in the dipole approximation, it cannot decay directly to the ground state $|111\rangle$. Show, in contrast, that it *can* decay directly to the state $|112\rangle$ and thence to the ground state $|111\rangle$.
- (b) [10] Calculate the total decay rate in these two cases.

Definite Integrals:

For each of the following formulas, m , n and p are assumed to be positive integers.

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right) dx = \begin{cases} 2a/n\pi & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

$$\int_0^a \cos\left(\frac{\pi nx}{a}\right) dx = 0$$

$$\int_0^a \cos\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) dx = \int_0^a \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) dx = \frac{1}{2} a \delta_{nm}$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) dx = \begin{cases} 2na/\pi(n^2 - m^2) & \text{if } n + m \text{ is odd} \\ 0 & \text{if } n + m \text{ is even} \end{cases}$$

$$\int_0^a \cos\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) \cos\left(\frac{\pi px}{a}\right) dx = \frac{1}{4} a (\delta_{n+m,p} + \delta_{n+p,m} + \delta_{p+m,n})$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) \cos\left(\frac{\pi px}{a}\right) dx = \begin{cases} \frac{2na(m^2 + p^2 - n^2)}{\pi[4m^2 p^2 - (m^2 + p^2 - n^2)^2]} & \begin{array}{l} n + m + p \\ \text{odd} \end{array} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) \cos\left(\frac{\pi px}{a}\right) dx = \frac{1}{4} a (-\delta_{n+m,p} + \delta_{n+p,m} + \delta_{p+m,n})$$

$$\int_0^a \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi px}{a}\right) dx = \begin{cases} \frac{4nmpa}{\pi[4m^2 p^2 - (m^2 + p^2 - n^2)^2]} & \begin{array}{l} n + m + p \\ \text{odd} \end{array} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^a x \sin\left(\frac{\pi nx}{a}\right) dx = -\frac{(-1)^n a^2}{\pi n},$$

$$\int_0^a x \cos\left(\frac{\pi nx}{a}\right) dx = \begin{cases} -2a^2/\pi^2 n^2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

$$\int_0^a x \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) dx = \begin{cases} a^2/4 & \text{if } n = m, \\ 2a^2 nm [(-1)^{n+m} - 1] / \pi^2 (n^2 - m^2)^2 & \text{if } n \neq m, \end{cases}$$

$$\int_0^a x \cos\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) dx = \begin{cases} a^2/4 & \text{if } n = m, \\ a^2 (n^2 + m^2) [(-1)^{n+m} - 1] / \pi^2 (n^2 - m^2)^2 & \text{if } n \neq m. \end{cases}$$