Physics 744 - Field Theory

Solution Set 5

All three of these problems deal with the $\psi^*\psi\phi$ theory, containing a complex field ψ and a real field ϕ, with Lagrangian density

$$\mathcal{L} = \partial_\mu \psi^* \partial_\mu \psi + \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - m^2 \psi^* \psi - \tfrac{1}{2} M^2 \phi^2 - \gamma \psi^* \psi \phi$$

Note that for all but part of problem 1, the interaction term is irrelevant.

1. [10] For this problem, treat the fields completely classically.

(a) [5] Write out the equations of motion for ψ, ψ^*, and ϕ. Verify that two of them are merely complex conjugates of each other.

This is straightforward. We have

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \quad \Rightarrow \quad -m^2 \psi^* - \gamma \psi^* \phi = \partial_\mu \partial^\mu \psi^*,$$

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} \quad \Rightarrow \quad -m^2 \psi - \gamma \psi \phi = \partial_\mu \partial^\mu \psi,$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \quad \Rightarrow \quad -M^2 \phi - \gamma \psi^* \psi = \partial_\mu \partial^\mu \phi.$$

Obviously, the first two are complex conjugates of each other.

(b) [5] Verify that $\psi \to e^{-i\theta}\psi$ is a symmetry of the theory. Work out the corresponding conserved current J_μ.

If $\psi \to e^{-i\theta}\psi$ then $\psi^* \to e^{i\theta}\psi^*$. It is obvious that if you make this substitution, the phases will cancel. The corresponding current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(-i\psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}(i\psi^*) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(0) = -i\psi \partial^\mu \psi^* + i\psi^* \partial^\mu \psi.$$

I don't know why I asked for it with one index down, but you can trivially lower it if you want.

2. **[10] We now want to quantize the theory in the interaction picture.**

(a) **[7] Write the conserved quantity $Q=\int J^0(\vec{x})d^3\vec{x}$ in terms of the annihilation operators $\alpha_{\vec{k}}$, $\beta_{\vec{k}}$, and $\gamma_{\vec{k}}$ and their corresponding creation operators. For consistency, let $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ annihilate the particle ψ and its corresponding anti-particle ψ^*, and let $\gamma_{\vec{k}}$ annihilate ϕ.**

The fields are given by

$$\psi(\mathbf{x})=\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left(\alpha_{\vec{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}+\beta_{\vec{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right),\quad \phi(\mathbf{x})=\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left(\gamma_{\vec{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}+\gamma_{\vec{k}}^\dagger e^{i\mathbf{k}\cdot\mathbf{x}}\right),$$

and $\psi^*(\mathbf{x})$ being given simply by the Hermitian conjugate of $\psi(\mathbf{x})$. We therefore have

$$\begin{aligned}
Q&=\int J^0(\mathbf{x})d^3\vec{x}=\int i\left[\psi^*(\mathbf{x})\partial^0\psi(\mathbf{x})-\psi(\mathbf{x})\partial^0\psi^*(\mathbf{x})\right]d^3\vec{x}\\
&=\int id^3\vec{x}\int\frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6 4\omega_k\omega_{k'}}\left\{\begin{array}{l}\left(\beta_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}-i\omega_{k'}t}+\alpha_{\vec{k}'}^\dagger e^{-i\vec{k}'\cdot\vec{x}+i\omega_{k'}t}\right)\left(-i\omega_k\alpha_{\vec{k}}e^{i\vec{k}\cdot\vec{x}-i\omega_k t}+i\omega_k\beta_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right)\\-\left(\alpha_{\vec{k}}e^{i\vec{k}\cdot\vec{x}-i\omega_k t}+\beta_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right)\left(-i\omega_{k'}\beta_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}-i\omega_{k'}t}+i\omega_{k'}\alpha_{\vec{k}'}^\dagger e^{-i\vec{k}'\cdot\vec{x}+i\omega_{k'}t}\right)\end{array}\right\}\\
&=\int\frac{d^3\vec{k}d^3\vec{k}'(2\pi)^3}{(2\pi)^6 4\omega_k\omega_{k'}}\left\{\begin{array}{l}\delta^3\left(\vec{k}-\vec{k}'\right)\left[\begin{array}{l}-\left(\omega_k\beta_{\vec{k}'}\beta_{\vec{k}}^\dagger+\omega_{k'}\beta_{\vec{k}}^\dagger\beta_{\vec{k}'}\right)e^{i(\omega_k-\omega_{k'})t}\\+\left(\omega_k\alpha_{\vec{k}'}^\dagger\alpha_{\vec{k}}+\omega_{k'}\alpha_{\vec{k}}\alpha_{\vec{k}'}^\dagger\right)e^{i(\omega_{k'}-\omega_k)t}\end{array}\right]\\+\delta^3\left(\vec{k}+\vec{k}'\right)\left[\begin{array}{l}\left(\omega_{k'}\beta_{\vec{k}}^\dagger\alpha_{\vec{k}'}^\dagger-\omega_k\alpha_{\vec{k}'}^\dagger\beta_{\vec{k}}^\dagger\right)e^{i(\omega_k+\omega_{k'})t}\\+\left(\omega_{k'}\alpha_{\vec{k}}\beta_{\vec{k}'}-\omega_k\beta_{\vec{k}'}\alpha_{\vec{k}}\right)e^{-i(\omega_k+\omega_{k'})t}\end{array}\right]\end{array}\right\}\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3 4\omega_k}\left\{\begin{array}{l}\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}+\alpha_{\vec{k}}\alpha_{\vec{k}}^\dagger-\beta_{\vec{k}}\beta_{\vec{k}}^\dagger-\beta_{\vec{k}}^\dagger\beta_{\vec{k}}\\+\left(\beta_{\vec{k}}^\dagger\alpha_{-\vec{k}}^\dagger-\alpha_{-\vec{k}}^\dagger\beta_{\vec{k}}^\dagger\right)e^{2i\omega_k t}+\left(\alpha_{\vec{k}}\beta_{-\vec{k}}-\beta_{-\vec{k}}\alpha_{\vec{k}}\right)e^{-2i\omega_k t}\end{array}\right\}\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3 4\omega_k}\left\{2\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}+\left[\alpha_{\vec{k}},\alpha_{\vec{k}}^\dagger\right]-\left[\beta_{\vec{k}},\beta_{\vec{k}}^\dagger\right]-2\beta_{\vec{k}}^\dagger\beta_{\vec{k}}\right\}=\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left\{\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}-\beta_{\vec{k}}^\dagger\beta_{\vec{k}}\right\}
\end{aligned}$$

(b) **[3] Write Q in terms of normalized, non-relativistic creation and annihilation operators, $a_{\vec{k}}$, $b_{\vec{k}}$ and $c_{\vec{k}}$. What is the total charge Q for a system containing n ψ's, m ψ^*'s and p ϕ's?**

Switching to finite volume notation

$$Q=\sum_{\vec{k}}\frac{1}{2V\omega_k}\left\{\alpha_{\vec{k}}^\dagger\alpha_{\vec{k}}-\beta_{\vec{k}}^\dagger\beta_{\vec{k}}\right\}=\sum_{\vec{k}}\left\{a_{\vec{k}}^\dagger a_{\vec{k}}-b_{\vec{k}}^\dagger b_{\vec{k}}\right\}$$

This is just the number operator for the ψ's and ψ^*'s, so if we have *n* and *m* of these respectively, then $Q=n-m$.

3. [15] Work out expressions for all six of the free propagators given below, and write the answer in a manifestly Lorentz invariant manner (so it has an $\int d^4\mathbf{k}$ and a $\lim_{\varepsilon\to 0}$, as in class). Most of them will be trivially zero.

$$\langle 0|\mathcal{T}\left[\phi(\mathbf{x})\phi(\mathbf{y})\right]|0\rangle,\quad \langle 0|\mathcal{T}\left[\psi(\mathbf{x})\phi(\mathbf{y})\right]|0\rangle,\quad \langle 0|\mathcal{T}\left[\psi^*(\mathbf{x})\phi(\mathbf{y})\right]|0\rangle,$$
$$\langle 0|\mathcal{T}\left[\psi(\mathbf{x})\psi(\mathbf{y})\right]|0\rangle,\quad \langle 0|\mathcal{T}\left[\psi^*(\mathbf{x})\psi^*(\mathbf{y})\right]|0\rangle,\quad \langle 0|\mathcal{T}\left[\psi^*(\mathbf{x})\psi(\mathbf{y})\right]|0\rangle.$$

Any term in these expressions will contain either two annihilation operators, two creation operators, or a creation and an annihilation operator. It will vanish if there is an annihilation operator on the right or a creation operator on the left. Therefore, the only non-vanishing terms will be those with a creation operator on the right and an annihilation operator on the left. Furthermore, if these operators commute, then we will get zero, so it can be non-vanishing only if it is the *same* creation and annihilation operators. Looking at our operators, we quickly realize the only pairs that have matching operators are the propagators $\langle 0|\mathcal{T}\left[\phi(\mathbf{x})\phi(\mathbf{y})\right]|0\rangle$ and $\langle 0|\mathcal{T}\left[\psi^*(\mathbf{x})\psi(\mathbf{y})\right]|0\rangle$.

To work out the first of these, write everything out explicitly. Assume first that $x^0 > y^0$, then we have

$$\begin{aligned}\langle 0|\phi(\mathbf{x})\phi(\mathbf{y})|0\rangle &= \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}\langle 0|\left(\gamma_{\vec{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}+\gamma^\dagger_{\vec{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right)\left(\gamma_{\vec{k}'}e^{-i\mathbf{k}'\cdot\mathbf{y}}+\gamma^\dagger_{\vec{k}'}e^{i\mathbf{k}'\cdot\mathbf{y}}\right)|0\rangle \\ &= \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}e^{i\mathbf{k}'\cdot\mathbf{y}-i\mathbf{k}\cdot\mathbf{x}}\langle 0|\left(\left[\gamma_{\vec{k}},\gamma^\dagger_{\vec{k}'}\right]+\gamma^\dagger_{\vec{k}'}\gamma_{\vec{k}}\right)|0\rangle \\ &= \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}e^{i\mathbf{k}'\cdot\mathbf{y}-i\mathbf{k}\cdot\mathbf{x}}(2\pi)^3(2\omega_k)\delta^3\left(\vec{k}-\vec{k}'\right) = \int\frac{e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}d^3\vec{k}}{(2\pi)^3 2\omega_k}\end{aligned}$$

This is the result if $x^0 > y^0$. In the other case, of course, we simply interchange the roles of **x** and **y**. As in class, we then proceed to write this in a more manifestly Lorentz invariant fashion,

$$\langle 0|\mathcal{T}\left[\phi(\mathbf{x})\phi(\mathbf{y})\right]|0\rangle = \lim_{\varepsilon\to 0^+}\int\frac{d^4\mathbf{k}}{(2\pi)^4}\frac{ie^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}}{\left(\mathbf{k}^2-M^2+i\varepsilon\right)}$$

We now need to repeat this for our other expression. Again assuming $x^0 > y^0$, we have

$$\begin{aligned}\langle 0|\psi^*(\mathbf{x})\psi(\mathbf{y})|0\rangle &= \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}\langle 0|\left(\beta_{\vec{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}+\alpha^\dagger_{\vec{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right)\left(\alpha_{\vec{k}'}e^{-i\mathbf{k}'\cdot\mathbf{y}}+\beta^\dagger_{\vec{k}'}e^{i\mathbf{k}'\cdot\mathbf{y}}\right)|0\rangle \\ &= \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}e^{i\mathbf{k}'\cdot\mathbf{y}-i\mathbf{k}\cdot\mathbf{x}}\langle 0|\left(\left[\beta_{\vec{k}},\beta^\dagger_{\vec{k}'}\right]+\beta^\dagger_{\vec{k}'}\beta_{\vec{k}}\right)|0\rangle = \int\frac{e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}d^3\vec{k}}{(2\pi)^3 2\omega_k}\end{aligned}$$

When $y^0 > x^0$ we have

$$\langle 0|\psi(\mathbf{y})\psi^*(\mathbf{x})|0\rangle = \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}\langle 0|\left(\alpha_{\vec{k}'}e^{-i\mathbf{k}'\cdot\mathbf{y}} + \beta^\dagger_{\vec{k}'}e^{i\mathbf{k}'\cdot\mathbf{y}}\right)\left(\beta_{\vec{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} + \alpha^\dagger_{\vec{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\right)|0\rangle$$

$$= \int \frac{d^3\vec{k}d^3\vec{k}'}{(2\pi)^6(4\omega_k\omega_{k'})}e^{-i\mathbf{k}'\cdot\mathbf{y}+i\mathbf{k}\cdot\mathbf{x}}\langle 0|\left(\left[\alpha_{\vec{k}'},\alpha^\dagger_{\vec{k}}\right] + \alpha^\dagger_{\vec{k}}\alpha_{\vec{k}'}\right)|0\rangle = \int \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3\vec{k}}{(2\pi)^3 2\omega_k}$$

Once again, the only effect of $y^0 > x^0$ is to interchange the role of **x** and **y**, so we again find

$$\langle 0|\psi^*(\mathbf{x})\psi(\mathbf{y})|0\rangle = \lim_{\varepsilon\to 0^+}\int \frac{d^4\mathbf{k}}{(2\pi)^4}\frac{ie^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}}{\left(\mathbf{k}^2 - m^2 + i\varepsilon\right)}$$

The only difference is that the relevant mass for these particles is m, not M.