Physics 744 - Field Theory

Homework Set 3

1. Consider a Lagrangian for two scalar fields ϕ_1 and ϕ_2, which has no more than two derivatives, and is no higher than quadratic order in the fields. The Lagrangian must be of the form

$$\mathcal{L} = \tfrac{1}{2} A \partial_\mu \phi_1 \partial^\mu \phi_1 + B \partial_\mu \phi_1 \partial^\mu \phi_2 + \tfrac{1}{2} C \partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2)$$

   a) Define ϕ_1' by the equation $\phi_1 = \phi_1' - B\phi_2 / A$. Show that the kinetic term in $\mathcal{L}$ when rewritten in terms of ϕ_1' and ϕ_2 have the exact same form, except that $B = 0$. So without loss of generality, we can assume $B = 0$.

   b) Now work out the Hamiltonian for this system. Argue that it is bounded below (never very negative) only if A and C are both positive and V is also bounded below. Argue that by rescaling the two fields, we can always make $A = C = +1$.

2. The Lagrangian of the previous problem has now been reduced to the form

$$\mathcal{L} = \tfrac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \tfrac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - V(\phi_1, \phi_2)$$

   a) Since the potential is no higher than quadratic, it must be of the form

$$V(\phi_1, \phi_2) = D + E\phi_1 + F\phi_2 + \tfrac{1}{2} A\phi_1^2 + B\phi_1\phi_2 + \tfrac{1}{2} C\phi_2^2$$

   Since this potential is bounded below, it must have a minimum somewhere. Argue that if we shift ϕ_1 and ϕ_2 by adding constants to them, so that the new minimum is at $\phi_1 = 0 = \phi_2$, two of the terms will automatically vanish. Also, explain why the D term is irrelevant.

   b) Consider the field transformation

$$\phi_1 = \phi_1' \cos\theta - \phi_2' \sin\theta$$

$$\phi_2 = \phi_1' \sin\theta + \phi_2' \cos\theta$$

   Convince yourself (and me) that the kinetic term is unchanged by this field transformation.

   c) Show that the same change of field definitions can simplify the potential. Specifically, show that we can make B vanish if we choose

$$\tan(2\theta) = \frac{2B}{A - C}$$

3. Two fields ϕ_1 and ϕ_2 interact via a Lagrangian of the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\phi_1^2 + \phi_2^2\right)$$

where V is an arbitrary function. We will be considering the symmetry

$$\phi_1'(x,\theta) = \phi_1(x)\cos\theta - \phi_2(x)\sin\theta$$

$$\phi_2'(x,\theta) = \phi_1(x)\sin\theta + \phi_2(x)\cos\theta$$

a) Show that this is a symmetry, and the Lagrangian is unchanged by this transformation. (technically, you should also check that $\theta = 0$ is the null transformation).

b) Derive an expression for the corresponding conserved current.

c) Let ϕ be the complex field defined by $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Rewrite the Lagrangian density $\mathcal{L}$ in terms of ϕ and ϕ^*.

d) Rewrite the transformation above in the form $\phi'(x) = f(\theta)\phi(x)$. What is the function f? Verify directly that the transformation leaves $\mathcal{L}$ unchanged in terms of this notation.

e) A naïve expression for the current density in terms of ϕ would be

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\frac{\partial\phi'}{\partial\theta}\bigg|_{\theta=0} + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^*)}\frac{\partial\phi'^*}{\partial\theta}\bigg|_{\theta=0}$$

Show that this naïve expectation is correct; that is, it leads to exactly the same current you found in part (b).

4. A real field has Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\gamma\phi^4$$

Consider the "scale invariance" symmetry,

$$\phi'(x,\lambda) = e^\lambda\phi\left(xe^\lambda\right)$$

a) Convince yourself (and me) that

$$\frac{d}{d\lambda}\phi'(x,\lambda)\bigg|_{\lambda=0} = \phi(x) + x^\nu\partial_\nu\phi(x)$$

and

$$\partial_\mu\frac{d}{d\lambda}\phi'(x,\lambda)\bigg|_{\lambda=0} = 2\partial_\mu\phi(x) + x^\nu\partial_\nu\partial_\mu\phi(x)$$

b) Show that

$$\frac{d}{d\lambda}\mathcal{L}(\phi',\partial_\mu\phi')\bigg|_{\lambda=0} = 4\mathcal{L} + x^\nu\partial_\nu\mathcal{L} = \partial_\nu\left(x^\nu\mathcal{L}\right)$$

but only if one of the terms in the Lagrangian vanishes. Hence this is a symmetry only if one of the terms is zero. Which one? (note that this statement is only true in four space-time dimensions).