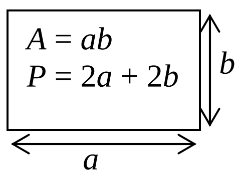
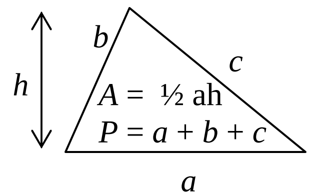
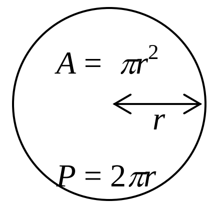
Math Review

Geometry

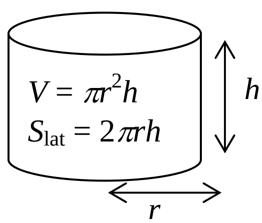
Much of what you learned in high school geometry is either pretty intuitive or will not be needed for this class, but many formulas you learned earlier for area or volume will come up a lot. Three two-dimensional shapes come up a lot: rectangles, triangles, and circles. For each of these shapes, the perimeter P is the distance around it, and the area A is the total size of the content. For polygons like the triangle and the rectangle, the perimeter is just the sum of the sides, while for the circle it is 2π times the radius.

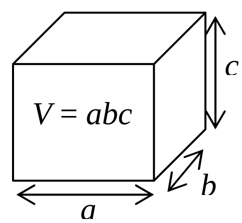






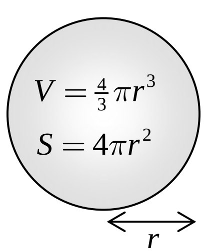
We will also be working with three-dimensional objects. A lot of our shapes will be generalized cylinders, which are formed by taking any two-dimensional shape and stretching it vertically. These objects have a volume that is equal to their base area times their height, $V = Ah$, and a lateral surface area $S_{\text{lat}} = Ph$; however, they also have additional surface areas at each end. The cases we will most commonly encounter are rectangular prisms (boxes) and circular cylinders, for which the formulas below apply.

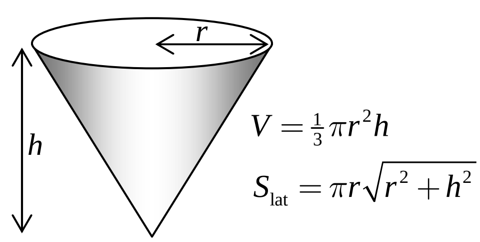




The cylinder will have an additional surface area of πr^2 on each end, and the surface area on any face of the box can be calculated from the rectangle area formula. For a cube of size a, the volume is $V = a^3$ and the total surface area is $S = 6a^2$.

One of the most common shapes we will encounter is a sphere, and for this you simply memorize the formula for the volume and surface area. We may also occasionally encounter a cone, for which the formula for the volume is simple, but the lateral surface area is kind of complicated.





For the cone, there is an additional surface area of πr^2 coming from the flat surface.

Algebra

Most of algebra is pretty straightforward. One thing that comes up fairly often in physics is the quadratic formula; that is, finding the solutions of the equation:

$$ax^2 + bx + c = 0$$

This has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

One other common expression that comes up is

$$\sqrt{x^2} = |x|,$$

not *x*.

Powers and exponents come up a lot. Some trivial formulas that you should remember are

$$x^0 = 1, \qquad x^1 = x, \qquad x^{1/2} = \sqrt{x} \qquad \text{and} \qquad x^{-n} = \frac{1}{x^n}$$

The following rules for combining exponentials are also helpful:

$$x^n x^m = x^{n+m}, \quad \frac{x^n}{x^m} = x^{n-m} \qquad \text{and} \qquad \left(x^n\right)^m = x^{nm}$$

Very often, especially when working with calculus, we get expressions with *e* raised to various powers, where *e* is the base of the natural logarithm. The three rules above apply in particular to e^x:

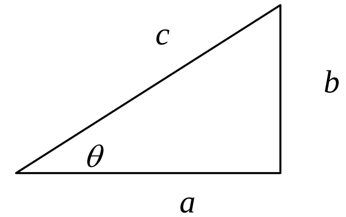
$$e^x e^y = e^{x+y}, \quad \frac{e^x}{e^y} = e^{x-y} \qquad \text{and} \qquad \left(e^x\right)^y = e^{xy}$$

One other useful fact to remember is that the inverse function of e^x is the natural logarithm ln*x*, so we have

$$e^x = y \Leftrightarrow x = \ln y$$

Trigonometry

In trigonometry, we very often need to find the angle and hypotenuse in a right triangle from the legs, or alternatively, given the angle and hypotenuse, we need to find the legs. The following formulas can help find these quickly:



$$c^2 = a^2 + b^2 \qquad \sin\theta = \frac{b}{c}$$

$$\tan\theta = \frac{b}{a} \qquad \text{or} \qquad \cos\theta = \frac{a}{c}$$

The three trigonometric relations are often memorized by the mnemonic "SOH-CAH-TOA", which means sine is opposite over hypotenuse, cosine is adjacent over hypotenuse, and tangent is opposite over adjacent. From these we can very easily prove the very useful identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{and} \qquad \tan\theta = \sin\theta/\cos\theta$$

For these three basic trigonometric functions, certain values come up often enough that it is helpful to know their values. Rather than memorizing all of these, it is easier to memorize the pattern for sine ($\sqrt{n}/2$ for n = 0, 1, 2, 3, 4), and then memorize that the cosine is the same thing backwards, and tangent is the ratio. In addition, if you remember that sine and tangent are odd functions, while cosine is even, you can get the values for the negatives of each of these:

angle		sin	cos	tan
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	∞

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x, \qquad \text{and} \qquad \tan(-x) = -\tan x$$

In addition to the three standard trigonometric functions above, there are three others:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \text{and} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

Some day you should memorize these as well, but they don't come up as often. We also often encounter the inverse functions, typically written as $\sin^{-1} x$ and so on. These can be evaluated with a calculator, or for certain special values, by using the table above; for example, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

In trigonometry, there are two standard ways of measuring angles: in degrees or radians. There are 360° in a circle and 2π radians. Degrees are most commonly used when you are talking about a physical angle. Radians are <u>always</u> used when you are working with calculus. Most calculators can be used to calculate using either of these units. Make sure to check which one your calculator is set for before you begin a calculation!

The sum of angles formula tends to come up a lot in physics, so let's lay it out for cosine and sine:

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

I don't expect you to memorize these, but I will use them occasionally when I need them. From these we can easily prove the double angle formulas as well:

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$
$$\sin(2x) = 2\sin x \cos x$$

Vectors

Vectors come up so often in physics that most physicists learn them in physics classes, not math. A vector is a quantity that has both a magnitude and a direction, like displacement, velocity, force, or acceleration. A quantity with a magnitude but no direction is called a scalar. In this class I will denote vectors by putting them in bold face ($\mathbf{v}$), though when I write it I normally draw some sort of arrow over it ($\vec{v}$). Scalars will be generally denoted by math italic font (s).

In three dimensions, a vector has three components:

$$\mathbf{v} = \left(v_x, v_y, v_z\right) \qquad \text{or} \qquad \mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}},$$

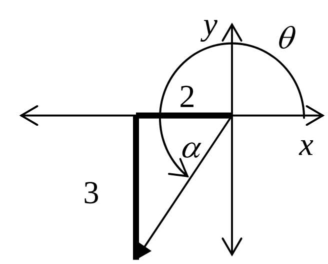
where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors in the *x*-, *y*-, and *z*-directions respectively. The little "roofs" over the symbols are read "hat" and signify a vector of unit length.

The length of a vector $\mathbf{v}$ can be determined using the 3D equivalent of the Pythagorean theorem. It is denoted by $|\mathbf{v}|$ or just plain *v*, and can be computed using

$$|\mathbf{v}| \equiv v \equiv \sqrt{v_x^2 + v_y^2 + v_z^2}\,.$$

It is very common in two dimensions to discuss the length *v* and the angle θ of a vector $\mathbf{v}$. The angle is normally measured counterclockwise starting from the *x*-axis. With the help of the trigonometric formulas above, it is not too difficult to convert from components to directions and vice versa. For example, suppose we were given the vector $\mathbf{v} = -2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$, and asked to compute the magnitude *v* and direction θ of this vector. The magnitude would be

$$v = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} = 3.606$$

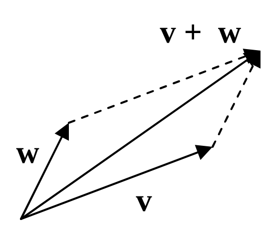


To find the direction, make a sketch of the vector as shown at right. The angle α at the origin can be seen to be

$$\alpha = \tan^{-1}(3/2) = 56.31°$$

The angle θ, however, should start from the +*x*-axis, and is therefore 180 degrees more than this, for a total of 236.31°.

Adding and subtracting vectors is pretty straightforward. To add two vectors, you simply make a little parallelogram out of them by copying each vector and placing its tail on the head of the other vector, as sketched at right. In components, the vectors can be added component by component, that is,



$$\mathbf{v} + \mathbf{w} = (v_x + w_x)\hat{\mathbf{i}} + (v_y + w_y)\hat{\mathbf{j}} + (v_z + w_z)\hat{\mathbf{k}}\,.$$

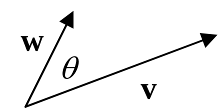
Subtracting vectors in component notation is similarly easy.

Multiplying with vectors is a bit more complicated. First of all, you can multiply (or divide) a vector $\mathbf{v}$ by a scalar s in a straightforward manner. Geometrically, $s\mathbf{v}$ points in the same direction as $\mathbf{v}$ but is *s* times longer. In components,

$$s\mathbf{v} = sv_x\hat{\mathbf{i}} + sv_y\hat{\mathbf{j}} + sv_z\hat{\mathbf{k}},$$

Such multiplication is commutative ($s\mathbf{v} \equiv \mathbf{v}s$), associative ($r(s\mathbf{v}) = (rs)\mathbf{v}$ and distributive ($(r+s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$ and $s(\mathbf{v}+\mathbf{w}) = s\mathbf{v} + s\mathbf{w}$).

Somewhat trickier is vector multiplication. It turns out there are two ways to multiply two vectors, called the dot and cross product, and it is important to keep them straight. The dot product is written $\mathbf{v}\cdot\mathbf{w}$ (pronounced "v dot w") and produces a scalar quantity, and is defined by



$$\mathbf{v}\cdot\mathbf{w} = vw\cos\theta,$$

where θ is the angle between the two vectors. In components, it is easy to calculate:

$$\mathbf{v}\cdot\mathbf{w} = v_x w_x + v_y w_y + v_z w_z .$$

The other way to multiply two vectors is called a cross product, written $\mathbf{v}\times\mathbf{w}$ (pronounced "v cross w") and produces a vector quantity. Like all vectors, it will have a magnitude and a direction. The magnitude is given by

$$|\mathbf{v}\times\mathbf{w}| = vw\sin\theta$$

The direction is defined to be perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, and chosen in accordance with the right-hand rule. The right-hand rule works as follows: Put your right hand out straight, but with your thumb pointed out, and make your fingers point in the direction of the vector $\mathbf{v}$. Now twist your wrist so that when you start to curl your fingers, your fingers will end up pointing in the direction $\mathbf{w}$. At this point, your thumb is pointing in the direction of $\mathbf{v}\times\mathbf{w}$. The only ambiguity occurs when $\mathbf{v}$ and $\mathbf{w}$ point in the same directions (parallel) or exactly opposite directions (anti-parallel), but in this case, $\sin\theta = 0$ and $\mathbf{v}\times\mathbf{w} = 0$. In the picture above, the cross-product points out of the paper.

In components, the cross product can be computed using the determinant:

$$\mathbf{v}\times\mathbf{w} = \det\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} = (v_y w_z - v_z w_y)\hat{\mathbf{i}} + (v_z w_x - v_x w_z)\hat{\mathbf{j}} + (v_x w_y - v_y w_x)\hat{\mathbf{k}}$$

It's messy, but occasionally it is useful.

When you combine dot or cross products with scalar multiplication or vector addition, it is easy to show that it is still associative and distributive, so that we have

$$(s\mathbf{v})\cdot\mathbf{w} = s(\mathbf{v}\cdot\mathbf{w}) \qquad (\mathbf{v}+\mathbf{w})\cdot\mathbf{x} = \mathbf{v}\cdot\mathbf{x} + \mathbf{w}\cdot\mathbf{x} \qquad \mathbf{x}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{x}\cdot\mathbf{v} + \mathbf{x}\cdot\mathbf{w}$$
$$(s\mathbf{v})\times\mathbf{w} = s(\mathbf{v}\times\mathbf{w}) \qquad (\mathbf{v}+\mathbf{w})\times\mathbf{x} = \mathbf{v}\times\mathbf{x} + \mathbf{w}\times\mathbf{x} \qquad \mathbf{x}\times(\mathbf{v}+\mathbf{w}) = \mathbf{x}\times\mathbf{v} + \mathbf{x}\times\mathbf{w}$$

Also, the dot product is commutative. However, the cross product is *anti*-commutative:

$$\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v} \quad \text{but} \qquad \mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}.$$

Triple vector products, involving three vectors combined with dot- and cross-products, are messier and we generally will not encounter them in this class.

Differentiation

Most of physics is written in terms of differential equations, and it is important to be able to take derivatives of even complicated functions quickly. Fortunately, it turns out that a few rules allow you to take the derivative of any function $f(x)$, no matter how complicated. The derivative of a function $f(x)$ is denoted either by

$$\frac{d}{dx} f(x) \equiv f'(x)$$

Higher derivatives can be denoted by similar notation,

$$\frac{d^2}{dx^2} f(x) \equiv f''(x), \quad \frac{d^3}{dx^3} f(x) \equiv f'''(x), \quad \text{etc.}$$

It is important to know the derivatives of a few simple functions, from which you can build up the derivatives of arbitrarily complicated functions. The basic functions you need to know to get derivatives of more complicated ones are (a and n are arbitrary real numbers):

$$\frac{d}{dx} a = 0, \qquad \frac{d}{dx} x^n = nx^{n-1}, \qquad \frac{d}{dx} e^x = e^x, \qquad \frac{d}{dx} \ln x = \frac{1}{x}.$$

You should memorize all four of these. In addition, the following derivatives of trigonometric and inverse trigonometric functions come up a lot:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2},$$

You should memorize at least the derivative of sine and cosine.

To take the derivative of more complicated functions, you need rules for taking the derivative of sums, differences, products and quotients of functions, as well as for functions of functions:

$$(f \pm g)' = f' \pm g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad [f(g)]' = f'(g)g'$$

Keep in mind that in mathematics, only the variable will be denoted by a letter like x, but in physics, there will generally be a lot of constants denoted by letters as well. As an illustration, let's find the derivative:

$$\frac{d}{dx}\left[Ne^{A\sin(kx^2+\phi)}\right] = N\left(\frac{d}{dx}e^{A\sin(kx^2+\phi)}\right) = Ne^{A\sin(kx^2+\phi)}\frac{d}{dx}\left(A\sin\left(kx^2+\phi\right)\right)$$

$$= -NAe^{A\sin(kx^2+\phi)}\cos\left(kx^2+\phi\right)\frac{d}{dx}\left(kx^2+\phi\right)$$

$$= -2NAkxe^{A\sin(kx^2+\phi)}\cos\left(kx^2+\phi\right)$$

Integration

There are, in fact, two different types of integration, called definite and indefinite integration. If $f(x)$ is an arbitrary function of x, then the indefinite integral $F(x)$ (sometimes called an anti-derivative) is defined to be that function F whose derivative is f, that is to say, $F' = f$. It is denoted by putting no limits on the integration symbol.

$$F(x) = \int f(x)\,dx \qquad \Leftrightarrow \qquad \frac{d}{dx}F(x) = f(x).$$

Because the derivative of a constant is zero, the indefinite integral F is defined only up to a constant, and hence in proper formalism the answer to an indefinite integration should always look like $F(x)+C$, where C is an unspecified constant of integration. Often this constant can be ignored. Make sure you keep the differential dx in your integration; if you ever change variables, this factor can be important!

A definite integral has limits of integration $x = a$ and b, and is defined as the area under the curve $f(x)$ starting from the point $x = a$ to the point $x = b$. The fundamental theorem of calculus relates the definite integral to the anti-derivative, namely,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b \equiv F(b) - F(a), \qquad \text{where} \quad F'(x) = f(x).$$

Because the definite integral involves the *difference* of F between the two endpoints, the constant of integration C always cancels out and is therefore irrelevant in a definite integral.

In contrast to differentiation, there are no simple rules to perform integration. Generally, you do your best to manipulate your integration into a relatively simple form, then you either immediately recognize the integral, or you look it up in an integral table (or better yet, learn how to use Maple to do integration for you). Many integrals cannot be written in simple closed form, in which case modern computers can numerically calculate the result, often to high accuracy, for most realistic problems.

A few rules that allow you to find indefinite integrals will help you. If a is an arbitrary constant, and $f(x)$ and $g(x)$ are functions whose integrals are $F(x)$ and $G(x)$ respectively, then it is not hard to show that

$$\int af(x)\,dx = a\int f(x)\,dx = aF(x),$$

$$\int f(x+a)\,dx = F(x+a),$$

$$\int \left[f(x) \pm g(x)\right]dx = \int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x).$$

Note also that it is commonly possible to do integration by substitution, namely, let x be any function of a new variable y, so $x = g(y)$, and then substitute this in. However, note that the differential dx transforms to $dx = d\big(g(y)\big) = g'(y)\,dy$, so the new integral becomes

$$\int f(x)\,dx = \int f\big(g(y)\big)\,d\big(g(y)\big) = \int f\big(g(y)\big)\,g'(y)\,dy$$

Integral Tables

The first four integrals on the left side, and perhaps the integrals of cosine and sine, should be memorized. For more complicated integrals, I recommend that you use Maple to assist you with the integral, or you can look it up in the following table. Note that all indefinite integrals have an implied + *C*, which can be ignored whenever you are performing a definite integral. In all expressions below, it is assumed that *a*, *b*, and *n* are real non-zero constants.

$$\int dx = x$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq 1$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax}$$

$$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax)$$

$$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax)$$

$$\int \tan(ax)\, dx = -\frac{1}{a}\ln|\cos(ax)|$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\, dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int \tan^2(ax)\, dx = \frac{\tan(ax)}{a} - x$$

$$\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x + \sqrt{x^2+a}\right|$$

$$\int \frac{dx}{\sqrt{a-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{a}}\right), \quad a > 0$$

$$\int \frac{dx}{x^2+a} = \frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right), \quad a > 0$$

$$\int \frac{dx}{x^2-a} = \frac{1}{2\sqrt{a}}\ln\left|\frac{x-\sqrt{a}}{x+\sqrt{a}}\right|, \quad a > 0$$

$$\int \frac{dx}{\left(a \pm x^2\right)^{3/2}} = \frac{x}{a\sqrt{a \pm x^2}}$$

$$\int \frac{x\,dx}{\sqrt{x^2+a}} = \sqrt{x^2+a}$$

$$\int \frac{x\,dx}{\sqrt{a-x^2}} = -\sqrt{a-x^2}, \quad a > 0$$

$$\int \frac{x\,dx}{x^2+a} = \tfrac{1}{2}\ln\left|x^2+a\right|$$

$$\int \frac{x\,dx}{\left(x^2+a\right)^{3/2}} = -\frac{1}{\sqrt{x^2+a}}$$

$$\int \frac{x\,dx}{\left(a-x^2\right)^{3/2}} = -\frac{1}{\sqrt{a-x^2}}, \quad a > 0$$

$$\int \frac{dx}{\left(x^2+a\right)\sqrt{x^2+b}} = \frac{1}{\sqrt{a(b-a)}}\sin^{-1}\left(\frac{x\sqrt{b-a}}{\sqrt{\left(x^2+a\right)b}}\right), \quad b > a > 0$$

$$\int x\sin(ax)\, dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax)$$

$$\int x\cos(ax)\, dx = \frac{x}{a}\sin(ax) + \frac{1}{a^2}\cos(ax)$$

Partial Derivatives and Multiple Integrations

In mathematics, it is common to work with only one variable, which we typically call x, but in physics it is common to have at least three dimensions (x, y, and z) and sometimes four (including time t). Hence quantities are commonly functions of several variables at once, we might write such a function as $f(x, y, z)$, for example. When differentiating, it is then common that we need to take derivatives in more than one direction, and in such cases we need a notation of *partial derivatives*. A partial derivative is just like an ordinary derivative, except we treat every variable except the one we are differentiating with respect to as a constant. For example, the partial derivative with respect to x is written as

$$\frac{\partial}{\partial x} f(x, y, z) \equiv \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

Just as with ordinary derivatives, we rarely use this definition, instead just using our ordinary rules for differentiating. For example, suppose we had the function $f(x, y, z) = A(2xy - z^2)$, and we were asked to take various partial derivatives of it. When taking the x derivative, for example, we would treat y and z as constants, and hence the first term would yield a derivative of $2Ay$, while the second term would yield nothing, since it is constant. So the three partial derivatives of this would be

$$\partial f/\partial x = 2Ay, \qquad \partial f/\partial y = 2Ax, \qquad \text{and} \qquad \partial f/\partial z = -2Az$$

It is also very common that in physics we need to perform a multiple-dimensional integration (these will always be definite integrals). In such circumstances, the integration should be worked from the inside out; that is, you first need to do the innermost integral, then work your way out to the outermost integral. One of the hardest parts of doing such an integration is setting it up in the first place, since, depending on the shape of the region you are integrating over, it may be very difficult to figure out the limits of integration. Often the limits of the inner integration will depend on the value of the variable in the outer integration. When doing the innermost integrations, all of the outer variables can be treated as constants.

Let's do an example to see how this works. Suppose we are faced with

$$\int_0^a dx \int_0^a dy \left(a^2 + x^2 + y^2\right).$$

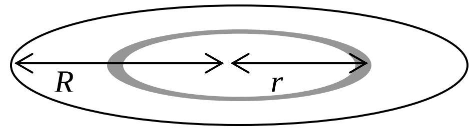
We start by performing the y integral (since that is inside). We treat a and x both as constants, and hence we have

$$\int_0^a dy \left(a^2 + x^2 + y^2\right) = \left(a^2 y + x^2 y + \tfrac{1}{3} y^3\right)\Big|_{y=0}^{y=a} = \tfrac{4}{3} a^3 + a x^2.$$

We then can easily finish the x-integration.

$$\int_0^a dx \int_0^a dy \left(a^2 + x^2 + y^2\right) = \int_0^a dx \left(\tfrac{4}{3} a^3 + a x^2\right) = \left(\tfrac{4}{3} a^3 x + \tfrac{1}{3} a x^3\right)\Big|_0^a = \tfrac{4}{3} a^4 + \tfrac{1}{3} a^4 = \tfrac{5}{3} a^4.$$

Many multi-dimensional integrals can be simplified greatly when there are symmetries involved. For example, suppose you have to perform an integration over a disk of radius R, but the integral is such that the integral depends only on the distance r from the center of the disk. The integration can be performed by dividing the disk into thin annuli, basically a slightly thickened circle, of radius r and thickness dr. The thin annulus can be thought of as a rectangle of length $2\pi r$ and width dr that has been bent into a circle, and therefore has area $2\pi r dr$. Hence the area differential dA can be replaced by $2\pi r dr$. For example, to calculate the integral of $1/\sqrt{r^2+a^2}$ over a disk, we would have



$$\int \frac{dA}{\sqrt{r^2+a^2}} = \int_0^R \frac{2\pi r dr}{\sqrt{r^2+a^2}} = 2\pi\sqrt{r^2+a^2}\Big|_0^R = 2\pi\left(\sqrt{R^2+a^2}-a\right)$$

This method can be used to calculate volume integrals for cylinders, cones, and spheres as well.

However, for spheres, the most common situation is one where you must perform an integration over a spherical volume, and the integral depends again only on the distance r from the center of the sphere. In this case, the most efficient way to do the computation is in terms of thin spherical shells, having a radius of r and thickness dr. The volume of this thin shell is the area of the shell ($4\pi r^2$) times the thickness, so we replace the volume element dV with $4\pi r^2 dr$. For example, if you are told that the total charge density of a sphere of radius R is given by $\rho(r) = Ar^2$, then the total charge of the sphere is

$$Q = \int \rho dV = \int_0^R \left(Ar^2\right) 4\pi r^2 dr = \frac{4\pi}{5} A r^5\Big|_0^R = \frac{4\pi}{5} AR^5$$