Solutions to Final Exam
December 12, 2008

This test consists of five parts. Please note that in parts II through V, you can skip one question of those offered. Some helpful equations and a short table of isotopes can be found on the last page

Part I: Multiple Choice (mixed new and review questions) [50 points]
For each question, choose the best answer (2 points each)

1. The purpose of giant interferometers that very carefully monitor the distance in two perpendicular directions (LIGO) is to
   A) Look for "frame dragging" effects caused by the rotation of the Earth
   B) Look for "time dilation" effects caused by time slowing down due to gravity
   C) Look for "precession of the perihelion" in Earth's orbit
   D) Look for "gravitational deflection of light" caused by light passing near the Sun
   E) Look for "gravity waves," distortions in space time caused by distant astronomical events

2. The central source of power for active galactic nuclei is generally believed to be
   A) A giant black hole in the center of the galaxy
   B) A giant star that recently went supernova
   C) Concentrated magnetic fields
   D) Matter-antimatter annihilation
   E) Relativistic electric currents

3. To find the probability of finding a particle with wave function $\psi(x)$ in the region $0 < x < a$, which of the following integrals should you perform?
   A) $\int_0^a \psi(x)\,dx$
   B) $\int_0^a \psi^2(x)\,dx$
   C) $\int_0^a \psi^*(x)\,dx$
   D) $\int_0^a |\psi(x)|\,dx$
   E) $\int_0^a |\psi(x)|^2\,dx$

4. According to the general theory of relativity, the orbit of Mercury around a source of gravity like the Sun should be:
   A) A circle that stays constant indefinitely
   B) A circle, but one that gradually shrinks towards zero over time
   C) An ellipse which it follows repeatedly
   D) Pretty close to an ellipse, but the long axis will slowly rotate (precess) around the Sun
   E) Pretty close to an ellipse, but the plane of the orbit will slowly wobble (nutate) up and down

5. If an object is moving in the *x*-direction, how will a stationary observer see its dimensions change?
   A) It will shrink, but only in the direction of motion
   B) It will expand, but only in the direction of motion
   C) It will shrink in all three dimensions
   D) It will expand in all three dimensions
   E) It will shrink in the direction of motion and expand in the other dimensions

6. The Schwarzschild solution, the formula that describes the metric around a black hole, can also be used
   A) For any spherically symmetric source of gravity, both inside and outside the object
   B) For any spherically symmetric source of gravity, but only outside the object
   C) For any spherically symmetric source of gravity, but only inside the object
   D) For neutron stars and other high mass objects, but not for planets or the Sun
   E) The Schwarzschild solution is useful only for black holes

7. The differential form of the distance formula, $c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ is superior to the form we originally learned, because
   A) You can use it to calculate the proper time for curved paths, not just straight ones
   B) It can be used at relativistic speeds, not just non-relativistic speeds
   C) In this form it already has curvature in it, an important part of general relativity
   D) It can be used for tachyons, not just slower-than-light particles
   E) It is smaller, and with today's economy, we all need to cut back

8. The main evidence for dark matter in spiral galaxies is
   A) Dust and other objects that obscure our view of the center
   B) High speed stars orbiting the center, indicating giant black holes
   C) Flat rotation curves showing that rotation velocity doesn't decrease as you get far from the center of a galaxy
   D) Shift in the wavelength of the 21 cm line indicating gravitational red shift from all the mass
   E) Huge clouds of cool hydrogen – "molecular clouds" – which is presumably the source of the gravity

9. Rutherford found that α-particles fired at very thin gold foil usually went almost straight through, but occasionally deflected at large angles, even straight backwards. From this he concluded that
   A) The alpha particle must be composed of a highly elastic material
   B) The electrons orbited the nucleus, rather than the other way around
   C) The atom must contain protons, in addition to electrons
   D) Most of the mass and positive charge of an atom must be concentrated in a small region he named the nucleus
   E) The alpha particle must be much lighter than the electrons, since they could bounce it backwards

10. What is the minimum and maximum energy that an object with a rest mass of m can have?
    A) 0, mc^2
    B) 0, infinity
    C) mc^2, infinity
    D) mc^2, mc^2 (i.e., it's always mc^2)
    E) None of the above

11. According to quantum mechanics, which types of objects listed below have wave properties?
    A) Particles of light (photons), but not electrons
    B) Electrons, but not photons
    C) Electrons and photons, but not atoms
    D) Electrons, photons, and atoms, but not molecules
    E) Electrons, photons, atoms, and molecules

12. The difference between wave velocity and group velocity is:
    A) Wave velocity is how fast the peaks move, group velocity is how fast the troughs move
    B) Wave velocity is how fast the troughs move, group velocity is how fast the peaks move
    C) Wave velocity is how fast the the individual peaks and troughs move, and group velocity is how fast whole wave as a "lump" moves
    D) Wave velocity is how fast the whole wave as a "lump" moves, and group velocity is how fast the individual peaks and troughs move
    E) Wave velocity and group velocity are different names for the same thing

13. In special relativity, which of the following is an accurate comment about conservation of momentum?
    A) It still always applies, and in exactly the same way it did before special relativity
    B) It still always applies, but you have to use a different formula for momentum
    C) It only sometimes applies, but the formula for momentum is still the same
    D) It only sometimes applies, and you have to use a different formula for momentum
    E) There is no conservation of momentum rule in special relativity

14. Which of these might be the actual mass of a ^{209}Bi atom?
    A) 83.020 u B) 125.987 u **C) 208.980 u** D) 209.984 u E) 292.016 u

15. Which of the following accounts for the phenomenon of quantum tunneling, where a particle makes it through a barrier that is taller than the energy of the particle?
    A) Thermal fluctuations occasionally cause the particle to have extra energy, making it over the barrier
    B) Due to the uncertainty principle, the particle's position is uncertain, so occasionally it just winds up on the other side of the barrier
    C) Due to quantum fluctuations, the barrier height occasionally fluctuates low enough for the particle to jump over it
    D) The wave function in the classically forbidden region dies exponentially, which means the probability is small, but not zero
    E) Virtual particle-antiparticle pairs appear out of nowhere and give the particle the extra energy to jump the barrier, before disappearing

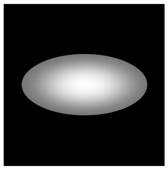
16. For heavy nuclei, the most stable nuclei tend to have about what fraction of neutrons?
    A) 90% **B) 60%** C) 50% D) 40% E) 10%

17. According to special relativity, how does the speed of light in vacuum differ as viewed by different observers?
    A) Only a special observer at rest will see it moving at *c*, all other observers will see it moving at different speeds
    B) Only observers moving at low velocities will see it moving at *c*, all other observers will see it moving at different speeds
    C) Only observers moving at relativistic velocities will see it moving at *c*, all other observers will see it moving at different speeds
    D) Because it is moving so fast, it is impossible for any observer to measure its speed
    E) All observers will see it moving at the same speed *c*

18. Which was an important aspect of the Bohr model of the atom that had not been considered previously?
    A) Photons come in only discrete chunks of energy
    B) Atoms can have electrons orbiting the nucleus in ellipses, not just circles
    C) The frequency of light coming from an atom is determined by the energy difference from the electron jumping from one level to another
    D) The frequency of light coming from an atom is determined by the frequency at which an electron orbits the atom
    E) Electrons should be described as waves, not just photons

19. How many neutrons are there in ^{42}Ca (element # 20)?
    A) 42 B) 20 C) 62 **D) 22** E) insufficient information

20. If a picture of a galaxy looked like the sketch at right, how should you most likely classify it?
A) Sb B) SBc C) E0 **D) E5** E) Irr



21. Which types of particles are affected by the strong force, or strong nuclear force?
A) Electrons (only)
B) Protons (only)
C) Neutrons (only)
D) Protons and electrons, but not neutrons
E) Protons and neutrons, but not electrons

22. In which of the following decays do you not get any neutrinos or anti-neutrinos out?
A) α-decay (only)
B) β^+-decay (only)
C) electron capture
D) α-decay and electron capture
E) α-decay and β^+-decay

23. The energy levels of the simple harmonic oscillator is given in back. If a particle in a simple harmonic oscillator moves from level $n = 3$ to level $n= 1$, how much energy would be released?
A) $\hbar\omega$ **B) $2\hbar\omega$** C) $3\hbar\omega$ D) $\frac{7}{2}\hbar\omega$ E) $5\hbar\omega$

24. Where in a galaxy like our own is one likely to find globular clusters?
A) The bulge **B) The halo** C) The disk D) The nucleus E) Hershey, PA

25. One version of the uncertainty principle relates which of the following two properties?
A) Momentum and position
B) Momentum and energy
C) Momentum and angular momentum
D) Energy and position
E) Energy and momentum

Part II: Short answer (review material) [20 points]

Choose **two** of the following three questions and give a short answer (1-3 sentences) (10 points each).

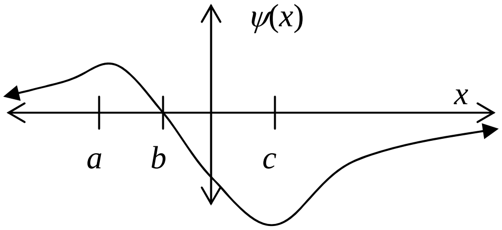
26. In a variation of the barn-pole paradox, a runner runs completely through the barn (in the front, out the back). The runner claims both ends of the pole simultaneously sticks out of the barn, while the farmer claims both ends of the pole are simultaneously inside the barn. Explain how special relativity helps resolve this apparent paradox.

The key to understanding this puzzle is simultaneity. To determine if the pole is inside the barn, you need to measure everything's position simultaneously, but different observers will disagree on what is simultaneous.

27. In the Compton effect, why is it that the wavelength of an X-ray changes when it scatters off of a stationary electron?

X-rays, like all particles of light (photons) carry a specific amount of energy, momentum, and so on. When an X-ray bounces off an electron, it transfers momentum to it, which means it is also giving it energy, and this means (by conservation of energy) that the photon loses energy, which means it has a lower frequency, which means it has a longer wavelength.

28. The graph at right illustrates a wave function. Explain where the particle is most likely to be and where it is least likely to be. You don't have to perform any computations.



The particle is most likely to be at a place where the wave function has the largest absolute value, which is at $x = c$ (the point $x = a$ represents a local maximum, but not a global maximum). It is least likely to be where the wave function has the smallest absolute value, which is at $x = b$, where it vanishes.

Part III: Short answer (new material) [30 points]

Choose **three** of the following four questions and give a short answer (1-3 sentences) (10 points each).

29. How does γ-decay change the value of Z and A? How dangerous is it, compared to other type of radiation? What types of nuclei undergo γ-decay?

Gamma decay changes neither Z nor A. Gamma radiation is more penetrating, and therefore more dangerous, than alpha or beta particles. The only types of nuclei that can gamma decay are those that are in an excited state, since a ground state nucleus can never lose energy without changing its identity.

30. How can we tell that the source of power in some active galaxies must be very small? Give any relevant formulas

Since information never travels more quickly than the speed of light, it is impossible for a large object to be "synchronized" and change all at once. Since some galaxies have power sources that fluctuate in a day or less, they must be no more than a light-day or so across. In particular, if something changes in time t, it can't be bigger than about $d = ct$ in size.

31. Explain qualitatively what a geodesic is. Under what circumstances, according to general relativity, do objects follow geodesics?

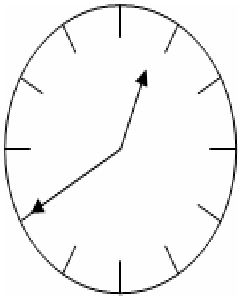
In ordinary three-dimensional space, a geodesic is the shortest path between two points, but in four-dimensional spacetime, it is the longest proper time path between two events. According to general relativity, objects follow geodesics provided they have no forces other than gravity acting on them. Indeed, gravity isn't exactly a force, it simply represents the curved paths objects take through curved spacetime when they follow geodesics.

32. Iron (Fe) is element 26. Would you expect ^{56}Fe to be a fairly stable nucleus, and why?

^{56}Fe contains 26 protons and 30 neutrons. These numbers are both even (which is good), about equal (good), and since iron is an intermediate mass (not very light/not very heavy) it has a small surplus of neutrons (good). All of these contribute to stability. It does not have any of the magic numbers that are supposed to make nuclei more stable; nonetheless, it is an extremely stable nucleus.

Part IV: Calculation (review material) [40 points]

Choose **two** of the following three questions and perform the indicated calculations (20 points each)



33. A circular clock is set to 12:00, and then accelerated to a constant high speed. Less than an hour later, it looks like the sketch at right, such that a stationary observer sees the clock having an *apparent* size of 10 cm tall and 8 cm wide, as sketched at right.

(a) Which way is the clock moving, *i.e.*, is it moving up/down or left/right? What is the actual size of the clock?

The clock is actually round, and therefore its distorted shape must be due to Lorentz contraction. Since this shrinks the clock in the direction of motion, the clock must be moving to the right or two the left. In the perpendicular (vertical direction), there is no distortion, so it must really be 10 cm tall, and since it is round, it is also 10 cm wide.

(b) What is γ for this clock, and what is the actual velocity?

The formula for Lorentz contraction is $L = L_p/\gamma$, so since L_p = 10 cm, and L = 8 cm, we have

$$\gamma = L_p/L = (10\text{ cm})/(8\text{ cm}) = \tfrac{5}{4} = 1.25$$

This can be related to velocity with the help of

$$\gamma = 1\big/\sqrt{1 - v^2/c^2}\,,$$

$$1 - v^2/c^2 = 1/\gamma^2 = \left(\tfrac{4}{5}\right)^2,$$

$$v/c = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - \tfrac{16}{25}} = \sqrt{\tfrac{9}{25}} = \tfrac{3}{5}$$

$$v = \tfrac{3}{5}c = 0.600\left(2.998\times10^8\text{ m/s}\right) = 1.799\times10^8\text{ m/s}$$

(c) You can tell how much time has passed from the sketch, according to the moving clock. According to an observer at rest, how many minutes have passed?

We know the proper time is 40 minutes, and from this we can calculate the time as viewed by us:

$$t = \gamma\tau = \tfrac{5}{4}(40\text{ min}) = 50\text{ min}$$

Metal	ϕ (eV)
Cs	2.14
K	2.29
Li	2.93
Mg	3.66
Sn	4.42

34. A laser produces light with a wavelength of 393.0 nm

(a) Find the frequency *f*, angular frequency *ω*, and wave number *k* for this wave.

The frequency is related to wavelength by $c = \lambda f$, so

$$f = \frac{c}{\lambda} = \frac{2.998\times10^{8}\text{ m/s}}{393.0\times10^{-9}\text{ m}} = 7.628\times10^{14}\text{ Hz}$$

The angular frequency is

$$\omega = 2\pi f = 2\pi\left(7.628\times10^{14}\text{ Hz}\right) = 4.79\times10^{15}\text{ s}^{-1}$$

and wave number by

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{393.0\times10^{-9}\text{ m}} = 1.599\times10^{7}\text{ m}^{-1}$$

(b) Find the energy (in eV) and momentum (in any units you want) for this photon.

The energy can most easily be found from

$$E = hf = \left(4.136\times10^{-15}\text{ eV}\cdot\text{s}\right)\left(7.628\times10^{14}\text{ s}^{-1}\right) = 3.155\text{ eV}$$

and the momentum from

$$p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\text{ J}\cdot\text{s}}{393.0\times10^{-9}\text{ m}} = 1.686\times10^{-27}\text{ kg}\cdot\text{m/s}$$

(c) The laser is shone on a variety of metals, listed at right. In which cases would electrons be emitted? In those cases where the electron is emitted, tell me the kinetic energy of the resulting electrons.

The light will hit the metal, and the work function is required to liberate the electron; the balance can then be used to produce kinetic energy for the electron. There isn't enough energy in the photon to liberate a photon from Mg nor Sn, but for the other cases, the leftover kinetic energy of the electron is:

$$E_e\left(\text{Cs}\right) = 3.15\text{ eV} - 2.14\text{ eV} = 1.01\text{ eV},$$
$$E_e\left(\text{K}\right) = 3.15\text{ eV} - 2.29\text{ eV} = 0.86\text{ eV},$$
$$E_e\left(\text{Li}\right) = 3.15\text{ eV} - 2.93\text{ eV} = 0.22\text{ eV}.$$

35. A Li^{++} ion has exactly one electron in it. This single electron is in the n = 3 state.

(a) What is the binding energy of this electron (in eV)?

Lithium is element 3 (which you can either read off the periodic table at the front of the room, or deduce from the fact that Li^{++} contains exactly one electron). Therefore its binding energy is

$$E_3 = -\frac{(13.6\text{ eV})Z^2}{n^2} = -\frac{(13.6\text{ eV})3^2}{3^2} = -13.6\text{ eV}$$

(b) The total angular momentum squared L^2 and z-component of the angular momentum L_z are measured. What are the possible outcomes? Explain any constraints relating these *i.e.* you might say something like "if L^2 is $-17\hbar^2$ then L_z is $\pm\sqrt{3}\hbar$ "

The total angular momentum quantum number l is a non-negative integer smaller than n, so l = 0, 1, or 2. The angular momentum around the z-axis quantum number m is an integer running from - l to +l. The formulas for these quantities are

$$L^2 = \hbar^2\left(l^2 + l\right) \quad \text{and} \quad L_z = \hbar m$$

L^2	L_z
0	0
$2\hbar^2$	$-\hbar, 0, \hbar$
$6\hbar^2$	$-2\hbar, -\hbar, 0, \hbar, 2\hbar$

A list of all possible values of L^2, and the corresponding possible values of L_z, can be found in the table at right.

(c) The total spin squared S^2 and z-component of the angular momentum S_z are measured. What are the possible outcomes? Explain any constraints relating these.

The total spin is always s = ½ for an electron, so that $S^2 = \hbar^2\left(s^2 + s\right) = \tfrac{3}{4}\hbar^2$. The z-component runs from – s to + s, so it takes on only the values $m_s = \pm\tfrac{1}{2}$, so $S_z = \pm\tfrac{1}{2}\hbar$. There are no restrictions in this case, since there is only one value for S^2.

(d) If the electron suddenly shifted to the n = 4 state, would energy be emitted or absorbed? How much energy (in eV)?

To go up a level, energy must be added or absorbed. The new energy is given by

$$E_4 = -\frac{(13.6\text{ eV})Z^2}{n^2} = -\frac{(13.6\text{ eV})3^2}{4^2} = -7.65\text{ eV}$$

The change in energy is therefore

$$\Delta E = 13.6\text{ eV} - 7.65\text{ eV} = 5.95\text{ eV}$$

Part V: Calculation (new material): [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each)

36. ^{137}Cs is an unstable isotope that decays with a half-life of 30.23 y (y = 3.156 × 10^7 s). It has been suggested that terrorists might include such an isotope in a "dirty bomb" designed to create high levels of radioactivity for an extended time. Suppose that a dirty bomb of ^{137}Cs is exploded which produces counts initially at the rate of 13 million counts per second.

(a) What is the decay constant λ in sec^{-1} for ^{137}Cs?

We use the relevant formula, which is

$$\lambda = \frac{\ln 2}{t_{1/2}} = \frac{0.6931}{30.23\text{ y}} = \frac{0.002293\text{ y}^{-1}}{3.156\times10^{7}\text{ s/y}} = 7.266\times10^{-10}\text{ s}^{-1}$$

(b) Approximately how many atoms of ^{137}Cs would be required?

The rate at which we see decays is $R = \lambda N$, so

$$N = \frac{R}{\lambda} = \frac{13\times10^{6}\text{ s}^{-1}}{7.266\times10^{-10}\text{ s}^{-1}} = 1.789\times10^{16}$$

(c) How much mass (in g) of ^{137}Cs would be required?

Each atom of ^{137}Cs has a mass very close to 137 u. Therefore the mass is

$$M = N\left(137\text{ u}\right) = \frac{\left(1.789\times10^{16}\right)\left(137\text{ u}\right)}{6.022\times10^{23}\text{ u/g}} = 4.070\times10^{-6}\text{ g} = 4.070\ \mu\text{g}$$

This isn't actually a terrifically dangerous quantity, but it's enough to make people nervous.

(d) How long will we have to wait until the decay rate drops to 1 million per second?

The decay rate at later times is given by

$$R = R_0 e^{-\lambda t}\ .$$

Rearranging this slightly, and then taking the natural logarithm and solving for *t*, we have

$$\begin{aligned} e^{\lambda t} &= R_0/R = \left(13\times10^{6}\text{ s}^{-1}\right)\Big/\left(10^{6}\text{ s}^{-1}\right) = 13, \\ \lambda t &= \ln 13, \\ t &= \ln 13/\lambda = 2.565\Big/\left(0.002293\text{ y}^{-1}\right) = 1129\text{ y} \end{aligned}$$

This is the other problem with radioactivity; it can remain for a long time.

37. Photocopied with the equation on the next page is a portion of Appendix A from the text. ^{210}Pb, even though it is highly unstable, exists naturally because it is produced by certain nuclear processes. The goal of this problem is to determine which process might *produce* ^{210}Pb. You might want to organize your answer into a table.

(a) What isotope is the parent if the process that produces ^{210}Pb is α-decay? β^- decay? β^+ decay? Electron capture?

mode	Parent	Q (MeV)	?
α	^{214}Po	7.836	yes
β^-	^{210}Tl	5.490	yes
β^+	^{210}Bi	-1.084	no
e.c.	^{210}Bi	-0.062	no

For α-decay, the value of Z decreases by two, and A by four. Hence they must have been two greater and four greater, respectively, beforeheand. Hence the parent was ^{214}Po.

For the other decays, A stays at 210. For β^- decay, the charge on the nucleus increases to make Pb, so it must have come from one lower charge, or ^{210}Tl. For β^+ decay and electron capture, the charge decreases by one, so it comes from ^{210}Bi.

(b) What is the Q-value for each of these processes?

For α-decay, we use the formula

$$Q(\alpha) = \left(M_P - M_D - M_{4He}\right)c^2 = \left(213.995177\text{ u} - 209.984163\text{ u} - 4.002602\text{ u}\right)c^2$$
$$= 0.008412\text{ u}c^2 = 0.008412\left(931.5\text{ MeV}\right) = 7.836\text{ MeV}$$

For β^- decay and electron capture, the formula is the same but the parent mass varies.

$$Q\left(\beta^-\right) = \left(M_P - M_D\right)c^2 = \left(209.990057\text{ u} - 209.984163\text{ u}\right)c^2$$
$$= 0.005894\text{ u}c^2 = 0.005894\left(931.5\text{ MeV}\right) = 5.490\text{ MeV}$$
$$Q\left(\text{e.c.}\right) = \left(M_P - M_D\right)c^2 = \left(209.984096\text{ u} - 209.984163\text{ u}\right)c^2$$
$$= -0.000067\text{ u}c^2 = -0.000067\left(931.5\text{ MeV}\right) = -0.062\text{ MeV}$$

For β^+ decay, you get the same value as for electron capture except you have to subtract the rest energy of two electrons

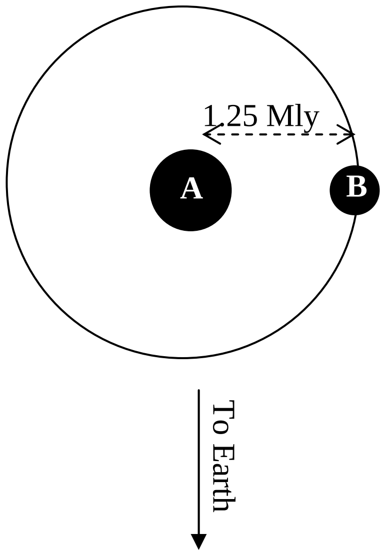
$$Q\left(\beta^+\right) = \left(M_P - M_D\right)c^2 - 2m_ec^2 = \left(209.984096\text{ u} - 209.984163\text{ u}\right)c^2 - 1.022\text{ MeV}$$
$$= -0.062\text{ MeV} - 1.022\text{ MeV} = -1.084\text{ MeV}$$

(c) Which processes are actually possible sources for ^{210}Pb?

Only those that produce positive values of Q are possible, so it can be made from α-decay of ^{214}Po and β^- decay of ^{210}Tl, but not from ^{210}Bi by either method.

38. A large central galaxy A has a smaller satellite galaxy B orbiting it at a distance of 1.25 Mly (1 Mly = 9.46 × 10^{21} m). The hydrogen-α line, normally at a wavelength of 656.28 nm, is observed to be at 678.35 nm coming from A and 678.02 nm coming from galaxy B.

(a) What is the approximate radial velocity of each of these galaxies towards or away from us, in km/s?



We first need the *z*-value for each of these, which are given by

$$z_A = \frac{\lambda_A - \lambda_0}{\lambda_0} = \frac{678.35\text{ nm} - 656.28\text{ nm}}{656.28\text{ nm}} = 0.03363$$
$$z_B = \frac{\lambda_B - \lambda_0}{\lambda_0} = \frac{678.02\text{ nm} - 656.28\text{ nm}}{656.28\text{ nm}} = 0.03313$$

Since these are both pretty small numbers, we can use the simple formula $z = v/c$ to estimate each of these velocities.

$$v_A = z_A c = 0.03363\left(299{,}800\text{ km/s}\right) = 10082\text{ km/s}$$
$$v_B = z_B c = 0.03313\left(299{,}800\text{ km/s}\right) = 9931\text{ km/s}$$

Since they came out positive, they are both moving away from us.

(b) Assume galaxy B is in circular orbit around galaxy A, such that we are viewing the orbit edge on (see sketch at right). Which direction, and at what velocity, is galaxy B orbiting?

Since galaxy *B* is moving more slowly away from us than galaxy *A*, it is in fact orbiting clockwise, or towards us at the moment. The orbital velocity is just

$$v = v_A - v_B = 10082\text{ km/s} - 9931\text{ km/s} = 151\text{ km/s}$$

(c) Estimate the mass of galaxy A, in solar masses (1 M_{Sun} = 1.989 × 10^{30} kg).

We take the formula for orbital velocity and rearrange it:

$$v = \sqrt{\frac{GM}{r}},$$
$$v^2 r = GM$$
$$M = \frac{v^2 r}{G} = \frac{\left(1.51\times10^5\text{ m/s}\right)^2\left(1.25\times9.46\times10^{21}\text{ m}\right)}{6.673\times10^{-11}\text{m}^3/\text{kg}/\text{s}^2} = \frac{4.04\times10^{42}\text{ kg}}{1.989\times10^{30}\text{ kg}/M_\odot}$$
$$= 2.03\times10^{12}\ M_\odot$$

39. A researcher is studying the black hole at the center of a galaxy with unknown mass. He spends two days, as measured by him, at a distance of 982 AU from the center of the black hole.

(a) When he rejoins his companions far from the black hole, he discovers that his clock is off from other clocks by exactly one hour. Will it be "slow" (i.e., behind them by an hour) or "fast" (i.e., ahead of them by an hour).

Since the effect of being in a gravitational well is to slow down time, the clock will be slow. Hence other clocks will show that time has advanced 49 hours, while his had only advanced 48 hours.

(b) What is the Schwarzschild radius for this black hole, in AU?

We use the formula

$$\tau = t\sqrt{1-\frac{2GM}{c^2 r}}$$

$$1-\frac{2GM}{c^2 r} = \frac{\tau^2}{t^2} = \frac{(48\ \mathrm{h})^2}{(49\ \mathrm{h})^2}$$

$$\frac{2GM}{c^2 r} = 1-\frac{48^2}{49^2} = 0.0404$$

$$R_S = \frac{2GM}{c^2} = 0.0404r = 0.0404(982\ \mathrm{AU}) = 39.7\ \mathrm{AU}$$

(c) What is the mass of the black hole, in solar masses? (1 AU = 1.496 × 10^{11} m, 1 M_{Sun} = 1.989 × 10^{30} kg).

We simply solve this equation for the mass *M*:

$$M = \frac{c^2 R_S}{2G} = \frac{\left(2.998\times10^{8}\ \mathrm{m/s}\right)^2 (39.7\ \mathrm{AU})\left(1.496\times10^{11}\ \mathrm{m/AU}\right)}{2\left(6.673\times10^{-11}\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2\right)}$$

$$= \frac{4.000\times10^{39}\ \mathrm{kg}}{1.989\times10^{30}\ \mathrm{kg}/M_\odot} = 2.011\times10^{9}\ M_\odot$$