Name ___________________

Solutions to Test 2
October 14, 2009

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered. The equations below may be helpful with some problems.

Constants

$$h = 6.626\times10^{-34}\ \text{J}\cdot\text{s} = 4.136\times10^{-15}\ \text{eV}\cdot\text{s}$$
$$\hbar = 1.055\times10^{-34}\ \text{J}\cdot\text{s} = 6.582\times10^{-16}\ \text{eV}\cdot\text{s}$$
$$k_B = 1.3807\times10^{-23}\ \text{J/K} = 8.6173\times10^{-5}\ \text{eV/K}$$
$$k = 8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2$$
$$e = 1.602\times10^{-19}\ \text{C}$$
$$\alpha = \frac{ke^2}{\hbar c} = 7.29735\times10^{-3} \simeq \frac{1}{137}$$

Black Bodies

$$U = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3}$$
$$\lambda_{\max} T = 2.898\times10^{-3}\ \text{m}\cdot\text{K}$$

Rutherford Scattering

$$b = \frac{kqQ}{m_\alpha v^2}\cot\left(\frac{\theta}{2}\right)$$
$$R = \frac{2Ze^2k}{E}$$

Compton Effect

$$\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$
$$\frac{h}{mc} = 2.426\times10^{-12}\ \text{m}$$

Wave Relationships

$$\lambda = \frac{2\pi}{k}$$
$$\frac{\omega}{2\pi} = f = \frac{1}{T}$$

Hydrogen-Like Atoms

$$E = -\frac{k^2e^4\mu Z^2}{2\hbar^2 n^2} = -\frac{(\mu c^2)\alpha^2 Z^2}{2n^2}$$
$$E = \frac{-(13.6\ \text{eV})Z^2}{n^2}$$

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. According to the particle data booklet, a highly unstable *W* boson requires 80.4 GeV of energy; however, it is not uncommon to produce one with about 2 GeV too little or too much energy. How can this be possible?
   A) Quantum mechanically, the creation of any particle involves some uncertainty in its position, so some of the energy may be coming from/going to an unknown source
   B) Any production of elementary particles is likely to be accompanied by additional incoming/outgoing photons, which carry additional energy
   C) Since it lives such a short time, there will be a large uncertainty in its energy, according to $\Delta E \Delta t \geq \frac{1}{2}\hbar$.
   D) Since one of the decay products of the *W* may be invisible, this particle is probably carrying off the missing energy
   E) Particle physics is incredibly difficult, and these must simply represent measurement errors

2. The Bohr model worked well on hydrogen, and other atoms with one electron, but not at all on most other atoms, with one notable exception, which was
   A) Helium atoms containing an extra electron
   B) The visible light spectrum from the outermost electrons of other atoms
   C) Noble gasses
   D) The scattering of X-rays from electrons of atoms
   E) The X-ray spectrum from the innermost electrons of other atoms

3. If the wave function for a particle is given by ψ, the probability density of finding the particle somewhere is given by
   A) ψ^* B) ψ^2 **C)** $\psi^*\psi$ D) $\left(\psi^*\right)^2$ E) None of these is correct

4. If $z = \sqrt{12+5i}$, then $|z|^2 =$
   A) 169 **B) 13** C) 7 D) $\sqrt{7}$ E) $\sqrt{119}$

5. The great contribution of deBroglie to quantum mechanics was
   A) The discovery of the electron
   B) The realization that electromagnetic energy comes in chunks, now called photons
   C) The suggestion that electrons might circle the nucleus, akin to a mini-solar system
   D) The suggestion that physical particles, such as electrons, might have wave properties
   E) The discovery of the nucleus

6. When Rutherford scattered alpha-particles off of very thin gold foil, he discovered, much to his surprise, that they sometimes would scatter at large angles, even backwards sometimes. From this he concluded
   A) The mass and positive charge must be concentrated in a tiny region he called the nucleus
   B) The positive charge and mass must be spread out through the atom, like plum pudding
   C) Alpha particles must be very light compared to atoms, rather than very heavy
   D) Electrons were actually heavier than alpha particles, and could deflect them a lot
   E) Many atoms must cluster into especially dense pockets, with large gaps between the atoms that were mostly empty space

7. Ordinary hydrogen ^1H and heavy hydrogen ^2H both have exactly the same nuclear charge, and identical electrons orbiting them, but heavy hydrogen has a nucleus that is twice as heavy. Why is it that the spectra are very slightly different for these two atoms?
   A) One nucleus is effectively a point, while the other is more spread out
   B) The magnetic interactions of the nucleus with the electron are different
   C) It is actually the nucleus, not the electron, that makes the transitions in heavy hydrogen
   D) The actual mass that comes into the computation of the energy is the "reduced mass", which depends (weakly) on the mass of the nucleus
   E) It is the interaction of the electron WITH the nucleus that makes it heavier, and hence also changes the spectrum

8. The importance of the Millikan oil drop experiment was that
   A) It was the first to demonstrate that oil, like water, was made of atoms
   B) It was the first demonstration that electrons actually have charge
   C) It was the first sign that electrons really did orbit the nucleus
   D) It was the first indication that charges come in chunks
   E) It allowed him to measure the elementary charge *e*.

9. In the Compton effect, X-rays that scatter from atoms come out with a slightly longer wavelength/lower frequency because
   A) The longer frequency is necessary to scatter coherently from different layers of atoms in crystals
   B) The X-ray photons have transferred energy into kinetic energy of the electrons, and the photons have lower energy and hence lower frequency
   C) The phase velocity for the photons have been converted to group velocity, which is lower
   D) The X-ray photons have been divided into pairs of photons, each of which has (naturally) less energy than the original
   E) Only integer numbers of wavelengths can scatter off of atoms

10. Another formula for $e^{i\theta}$ is
    A) $\cos\theta + i\sin\theta$
    B) $\cos\theta - i\sin\theta$
    C) $\sin\theta + i\cos\theta$
    D) $\sin\theta - i\cos\theta$
    E) None of these is correct

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (2-3 sentences) (10 points each).

11. What were the three assumptions Bohr used in trying to develop his model of the hydrogen atom?

He assumed (i) that the electrons orbit the nucleus, (ii) that the angular momentum of the electrons was always an integer multiple of $\hbar$, and (iii) when an electron went from one level to another, the atom absorbed or emitted a single photon of energy.

12. Under what circumstances does a lot of diffraction occur? In particular, how come you can easily hear around a corner, but you don't see around a corner, nor do material objects (like people) diffract around corners?

Diffraction occurs, or is large, when the wavelength of the object being diffracted is larger or comparable to the size of the aperture through which it is passing. Sound waves have wavelengths of the order of a meter (the size of a doorway or hallway), while light has wavelengths shorter than a micrometer, and a macroscopic object has a far, far smaller wavelength.

13. According to classical physics, the electron would prefer to have momentum zero and be exactly at the proton, where it would have infinite negative energy. Explain quantum mechanically why this doesn't happen. An equation or inequality might be in order.

According to quantum mechanics, it is impossible to control simultaneously both the position and momentum of a particle, hence we cannot demand both that the electron has no momentum and that it is at a particular position, since this would violate the uncertainty principle, $\Delta x \Delta p \geq \frac{1}{2}\hbar$. Hence the lowest energy state works out to be a compromise, where the atom has a finite size.

Part III: Calculation: [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each).

14. An important era of the early universe was *recombination*. At the time, the universe was at a temperature of approximately 2975 K, and was filled with a thermal distribution of radiation.

(a) [5] At what wavelength, in nm, would this radiation have been strongest?

The wavelength can be computed from Wien's Law, given among the equations, which says $\lambda_{\text{max}}T = 2.898\times10^{-3}\text{ m}\cdot\text{K}$, so that

$$\lambda_{\text{max}} = \frac{2.898\times10^{-3}\text{ m}\cdot\text{K}}{T} = \frac{2.898\times10^{-3}\text{ m}\cdot\text{K}}{2975\text{ K}} = 9.741\times10^{-7}\text{ m} = 974\text{ nm}$$

(b) [7] What would have been the energy of one typical photon at this wavelength?

The frequency is given by $f = c/\lambda$, and the energy by $E = hf = hc/\lambda$, so

$$E = \frac{hc}{\lambda} = \frac{\left(4.136\times10^{-15}\text{ eV}\cdot\text{s}\right)\left(2.998\times10^{8}\text{ m/s}\right)}{9.741\times10^{-7}\text{ m}} = 1.27\text{ eV}$$

(c) [8] What was the energy density, in J/m^3, at this time?

This is given by our other formula for black body radiation, namely,

$$U = \frac{\pi^2\left(k_BT\right)^4}{15\left(\hbar c\right)^3} = \frac{\pi^2\left[\left(1.3807\times10^{-23}\text{ J/K}\right)\left(2975\text{ K}\right)\right]^4}{15\left[\left(1.055\times10^{-34}\text{ J}\cdot\text{s}\right)\left(2.998\times10^{8}\text{ m/s}\right)\right]^3} = 0.0592\text{ J/m}^3$$

15. Nuclear forces require that a proton be inside the nucleus, a region typically 4 fm in radius ($4\text{ fm} = 4\times10^{-15}\text{ m}$).

(a) [5] If this is the radius, according to Carlson's rule, what is the uncertainty in its position?

Carlson's rule isn't a hard and fast rule, but in round numbers, if it is in a region of radius 4 fm, then it within a diameter of 8 fm, and we divide this by four to get an uncertainty in its position of $\Delta x = 2\text{ fm} = 2\times10^{-15}\text{ m}$.

(b) [7] What is the corresponding uncertainty in the momentum due to this motion?

By the uncertainty principle, $\Delta x \Delta p \geq \frac{1}{2}\hbar$, and therefore the proton has an uncertainty of at least

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.055\times10^{-34}\text{ J}\cdot\text{s}}{2\left(2\times10^{-15}\text{ m}\right)} = 2.64\times10^{-20}\text{ kg}\cdot\text{m/s}$$

$$= \frac{6.582\times10^{-16}\text{ eV}\cdot\text{s}}{2\left(2\times10^{-15}\text{ m}\right)}\cdot\frac{2.998\times10^{8}\text{ m/s}}{c} = \frac{4.93\times10^{7}\text{ eV}}{c} = 49.3\text{ MeV}/c$$

Either answer is fine.

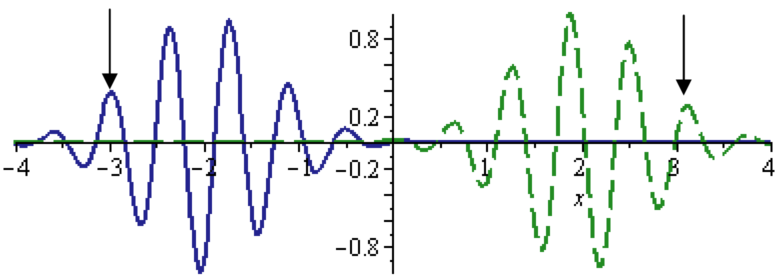
(c) [8] Predict the approximate kinetic energy of a proton, based on the fact that it is in the nucleus. Your answer may be in J or MeV. The proton has a mass of $m_p = 1.672\times10^{-27}\text{ kg} = 938\text{ MeV}/c^2$.

$$E = \frac{p^2}{2m} \approx \frac{(\Delta p)^2}{2m} = \frac{\left(2.64\times10^{-20}\text{ kg}\cdot\text{m/s}\right)^2}{2\left(1.672\times10^{-27}\text{ kg}\right)} = 2.084\times10^{-13}\text{ J}$$

$$\approx \frac{(\Delta p)^2}{2m} = \frac{\left(49.3\text{ MeV}/c\right)^2}{2\left(938\text{ MeV}/c^2\right)} = 1.30\text{ MeV}$$

These numbers turn out to be the same (not surprisingly).

16. A wave packet is observed to move from the initial position (solid lines) to the final position (dashed lines) at right over a period of time t = 12 s. The particular wave that starts at the first arrow is observed to move to the position of the final arrow. The x-axis is marked in meters.



(a) Estimate the wavelength λ of the wave (in m) and the wave number k.

Counting from the arrow on the left, it looks to me like there are three waves between x = -3 m and x = -1.2 m. The wavelength is therefore

$$\lambda = \frac{(-1.2\text{m}) - (-3.0\text{ m})}{3} = 0.6\text{ m}$$

The wave number is then

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.6\text{ m}} = 10.5\text{ m}^{-1}$$

I wrote the code, and know that the actual number is exactly 10, so we were off by about five percent.

(b) What is the phase velocity v_p and the group velocity v_g for this wave?

The phase velocity is governed by how fast the individual wave travels. It looks to me like it went from about -3 m to about 3.2 m, or a distance of 6.2 m, in about 12 seconds. For the group velocity, we see the center of the wave packet started around -2 m and travelled to 2 m in the same twelve seconds. So the two velocities are

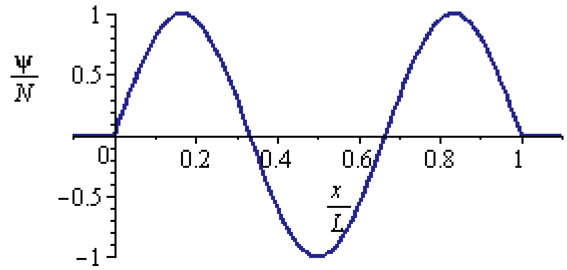
$$v_p = \frac{6.2\text{ m}}{12\text{ s}} = 0.52\text{ m/s} \qquad v_g = \frac{4.0\text{ m}}{12\text{ s}} = 0.33\text{ m/s}$$

(c) What is the angular frequency ω for this wave?

The phase velocity is in general given by the equation $v_p = \omega/k$. so

$$\omega = v_p k = (0.52\text{ m/s})(10.5\text{ m}^{-1}) = 5.5\text{ s}^{-1}.$$

17. One of the solutions of Schrödinger's time-independent equation for an infinite square well is



$$\psi(x) = \begin{cases} N\sin(3\pi x/L) & \text{if } 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

This wave function is sketched at right.

(a) In the region $0 < x < L$, is there any place that the particle cannot be?

The particle cannot be any place where the wave function vanishes. Since sin vanishes at integer multiples of pi, we have

$$\frac{3\pi x}{L} = \pi n, \quad \text{so} \quad x = \frac{nL}{3}$$

so it is at multiples of $\frac{1}{3}L$. The only ones in the region $0 < x < L$ are $x = \frac{1}{3}L$ and $x = \frac{2}{3}L$

(b) In the region $0 < x < L$, what is (are) the most likely place (places) to find the particle?

The most likely place to find the particle is when the wave function is the largest (positive or negative). This will occur when the derivative vanishes, which is when

$$0 = \frac{d\psi(x)}{dx} = N\frac{3\pi}{L}\cos\left(\frac{3\pi x}{L}\right).$$

The places where cosine vanishes is at half-integer multiples of pi, so we have

$$\frac{3\pi x}{L} = \pi\left(n + \tfrac{1}{2}\right), \quad \text{so} \quad x = \frac{L\left(n+\frac{1}{2}\right)}{3}$$

The only values where this lies within the allowed region are for n = 0, 1, and 2, so we have $x = \frac{1}{6}L$, $x = \frac{1}{2}L$, and $x = \frac{5}{6}L$. It isn't hard to see that at all of these the sine function takes on the values ± 1, so in fact they are all tied for being equally large.

(c) What is the correct normalization constant *N*? Some possibly useful integrals are given below.

The wave function is real, so the integral we need to satisfy is

$$1 = \int |\psi|^2\, dx = \int_0^L N^2 \sin^2\left(\frac{3\pi x}{L}\right) dx = \tfrac{1}{2}N^2 L$$

As found in class, this requires $N = \sqrt{2/L}$.

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 2L/n\pi & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \qquad \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = \tfrac{1}{2}L$$