Solutions for Test 2

Part I: Multiple Choice [20 points]

For each question, choose the best answer (2 points each)

1. Why is it more accurate to use the reduced mass μ rather than the electron mass m_e when calculating the binding energies of hydrogen-like atoms?
   A) The reduced mass takes into account the decrease in the electron mass due to relativistic effects
   B) The reduced mass takes into account the weakening of the electric force at high velocities
   C) The reduced mass includes the screening from other electrons in the atom
   D) The reduce mass includes the magnetic effects of the electron as well
   E) The reduced mass takes into account that it is not just the electron, but also the nucleus that moves as the electron orbits it

2. In the Franck Hertz experiment, as electrons are accelerated to higher and higher energies by increasing the voltage, at first the current increases, but then it decreases. What causes this decrease?
   A) The electrons have enough energy to bump the intervening atoms up to higher energy, so the electrons can now lose energy
   B) The energies are getting so high that the electrons tunnel through the collection plate, and hence don't contribute
   C) The electrons can only be collected if they have exactly the right energy when they hit the plate, and they now have too much
   D) The electrons contribute with so much current that the meter goes through a full circle to zero
   E) The electrons are starting to collide with each other, which slows them down

3. By measuring the curvature of electrons moving in electric and magnetic fields, J.J. Thomson was able to determine
   A) The mass of electrons
   B) The charge of electrons
   C) The size of electrons
   D) The ratio of charge to mass of electrons
   E) The ratio of mass to size of electrons

4. When a photon hits an electron at rest, and scatters at some non-zero angle, the final photon will have a
   A) Longer wavelength than the initial photon
   B) Shorter wavelength than the initial photon
   C) The same wavelength as the initial photon
   D) It can be either a longer or shorter wavelength, depending on the angle
   E) There is insufficient information to answer this question

5. In Wien's Law, $\lambda_{\max}T = 2.898\times10^{-3}\ \mathrm{m\cdot K}$, what does $\lambda_{\max}$ stand for?
   A) The longest wavelength that is produced in black-body radiation
   B) The wavelength where black-body radiation has the maximum power
   C) The maximum wavelength where the Planck distribution formula works
   D) The maximum wavelength where the Rayleigh-Jeans formula works
   E) The Maxwell wavelength, give by Maxwell's formulas for light

6. Which of the following objects is believed to have wave-like properties?
   A) Electrons (only)
   B) Protons (only)
   C) Atoms (only)
   D) Electrons and protons, but not atoms
   E) Electrons, protons, and atoms

7. If the complex number z is given by $z = e^{i\theta}$, what is $|z|^2$?
   A) $\cos^2\theta - i\sin^2\theta$ B) $\cos^2\theta - \sin^2\theta$ C) $e^{2i\theta}$ **D) 1** E) -1

8. What is the relationship between the Planck constant h and the reduced Planck constant $\hbar$?
   A) $h = \hbar/2\pi$ B) $h = \hbar/\pi$ **C) $\hbar = h/2\pi$** D) $\hbar = h/\pi$
   E) They are unrelated, they don't even have the same units.

9. Although the Bohr model only works really well for hydrogen (and hydrogen-like atoms), it works reasonably well in what other case?
   A) For the noble gasses
   B) For alkali metals, which have just one outer electron
   C) For X-ray spectra, produced by the innermost electrons of atoms (with some modification)
   D) For halogens, atoms that are missing just one electron in their outermost shell
   E) Actually, the Bohr model works very well for all of these

10. When Rutherford and his students scattered alpha-particles at the highest available energy, they found that the Rutherford formula did *not* work at the highest angles. This was because
   A) Relativistic corrections changed the results
   B) They were getting so close to the nucleus that they could see its finite size
   C) The energy was so large that the nucleus was moving as well, changing the results
   D) The alpha particles were shattering, changing their results
   E) The nuclei were shattering, changing their results

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (1-3 sentences) (10 points each).

11. What was the ultraviolet catastrophe? What did Planck have to assume to avert this catastrophe? Your answer should include an equation.

Before the work of Max Planck, it was assumed that light waves could have arbitrary energy, but this led to the (incorrect) prediction that most of the energy would be at the short-wavelength (ultraviolet) end of the spectrum, and even worse, that the total energy would be infinite. Planck assumed that the energy in electromagnetic waves could only come in multiples of $E = hf$, where f was the frequency of the waves and h was a new constant, now called the Planck constant.

12. Explain the difference between phase velocity and group velocity of a wave. If we had a formula $\omega = \omega(k)$ for the angular velocity as a function of wave number, how would we calculate each of these quantities?

Waves normally come in wave packets, which consist of a lot of little waves that together are lumped together into a big packet. The phase velocity is how fast each of the individual waves moves, while the group velocity is the velocity of the entire wave packet. If we have a dispersion relation, these two quantities can be calculated from the formulas

$$v_p = \frac{\omega(k)}{k} \qquad \text{and} \qquad v_g = \frac{d\omega(k)}{dk}.$$

13. What were the three assumptions Bohr had to make to correctly predict the energy levels of hydrogen?

Bohr assumed that the electron orbited in a circle around the nucleus, that the angular momentum had to come as an integer multiple of the reduced Planck constant $\hbar$, and that when the electron moved from one level to another, the energy difference came out as a single photon of energy $hf = \hbar\omega$. The last assumption turned out to be correct, though the first two weren't correct, though the Bohr model turned out to successfully predict the spectrum of hydrogen and hydrogen-like atoms.

Part III: Calculation: [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each).

14. A laser produces an ultrafast pulse of light with wavelength 475 nm. The pulse lasts only 15 fs (fs = 10^{-15} s) long with a total integrated power (total energy) of 0.100 J.

(a) What is the wave number *k*, angular frequency ω, and frequency *f* of this wave?

The wave number, frequency, and angular frequency can all be worked out from simple equations like $k\lambda = 2\pi$, $f\lambda = c$, and $\omega = ck$ to give

$$k = 2\pi/\lambda = 2\pi\big/\left(4.75\times10^{-7}\ \text{m}\right) = 1.32\times10^{7}\ \text{m}^{-1},$$
$$\omega = ck = \left(2.998\times10^{8}\ \text{m/s}\right)\left(1.32\times10^{7}\ \text{m}^{-1}\right) = 3.97\times10^{15}\ \text{s}^{-1},$$
$$f = c/\lambda = \left(2.998\times10^{8}\ \text{m/s}\right)\big/\left(4.75\times10^{-7}\ \text{m}\right) = 6.31\times10^{14}\ \text{s}^{-1}.$$

As a check, we note that $\omega = 2\pi f$, as it should.

(b) What is the approximate energy of each photon in Joules? How many photons in all were emitted?

The easiest way to do this is

$$E = hf = \left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)\left(6.31\times10^{14}\ \text{s}^{-1}\right) = 4.18\times10^{-19}\ \text{J}$$

Since the total amount of energy is 0.100 J, the number of photons is just

$$N = E_{\text{tot}}/E = hf = \left(0.100\ \text{J}\right)\big/\left(4.18\times10^{-19}\ \text{J}\right) = 2.39\times10^{17}$$

(c) Although lasers normally produce photons of only one frequency/energy/wavelength, an ultrashort pulse of this type will necessarily produce a spread in frequencies or energies. Estimate the uncertainty $\Delta\omega$ and ΔE in angular frequency and energy due to the short duration of the pulse.

Because the pulse lasted only 15 fs, there is an uncertainty in when it occurred (by Carlson's rule) of about 15 fs/4 = 3.75 fs. Using the classical and quantum uncertainty relations, $\Delta\omega\Delta t \ge \frac{1}{2}$ and $\Delta E\Delta t \ge \frac{1}{2}\hbar$, there is a corresponding uncertainty in the angular frequency and energy of

$$\Delta\omega \geq \frac{1}{2\Delta t} = \frac{1}{2\left(3.75\times10^{-15}\text{ s}\right)} = 1.33\times10^{14}\text{ s}^{-1},$$

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{1.055\times10^{-34}\text{ J}\cdot\text{s}}{2\left(3.75\times10^{-15}\text{ s}\right)} = 1.41\times10^{-21}\text{ J}.$$

This is about a 3.4% spread in energy or angular frequency.

15. A single electron is in an atom of unknown *Z* (so it is a hydrogen-like atom). It is found that when the electron falls from level *n* to level 1, it emits photons with energy 302 eV

(a) If we knew both the charge *Z* and the initial level *n*, find a formula for the change in energy. It may be useful to solve this equation for *Z*.

The energy of an electron in a hydrogen like atom is given by $E = -\left(13.6\text{ eV}\right)Z^2/n^2$, where Ze is the charge of the nucleus and n is the level of the electron. If it goes from level n to level 1, the photon that comes out will have an energy equal to the difference in energy, or

$$\Delta E = \left(13.6\text{ eV}\right)Z^2\left(1-\frac{1}{n^2}\right)$$

Equating this to the energy of the photon detected, we have

$$Z^2\left(1-\frac{1}{n^2}\right) = \frac{302\text{ eV}}{13.6\text{ eV}} = 22.2, \qquad \text{or} \qquad Z = \frac{4.71}{\sqrt{1-1/n^2}}$$

(b) If the initial level is *n* = 2, find the corresponding value of *Z*.

We simply plug in $n = 2$ in the previous equation to find

$$Z = \frac{4.71}{\sqrt{1-1/2^2}} = 5.44$$

(c) If the initial level is *n* = 3, find the corresponding value of *Z*.

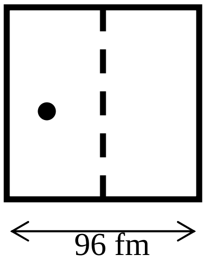
$$Z = \frac{4.71}{\sqrt{1-1/3^2}} = 5.00$$

(d) If the initial level is very large, say *n* = ∞, find the corresponding value of *Z*.

$$Z = \frac{4.71}{\sqrt{1-1/\infty^2}} = \frac{4.71}{\sqrt{1-0}} = 4.71$$

(e) What is the correct value of *Z* and *n*? Give an argument for your answer.

The value of both Z and n must both be an integer. It is clear from parts (c) and (d) that as you increase n from 3 to infinity, Z lies in the range from 5.00 to 4.71. But there are no integers in this range, so clearly n can't be in this range. Furthermore, $n = 2$ leads to a non-integer value for Z. Hence the only acceptable solution is the answer given in part (c), where we found $n = 3$ and $Z = 5$.



16. A single proton of mass $m = 1.673 \times 10^{-27}$ kg is confined in a small cubical box of length $L = 96.0$ fm (fm = 10^{-15}m), as illustrated at right (for now, ignore the dashed line). For this entire problem, I am asking for estimates only.

(a) What is the uncertainty in the position? What is the corresponding uncertainty in the momentum?

By Carlson's rule, the uncertainty in the position is ¼l = 24.0 fm. By the uncertainty principle, the corresponding uncertainty in the momentum is

$$\Delta p = \frac{\hbar}{2\Delta x} = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{2\left(24.0\times10^{-15}\ \text{m}\right)} = 2.20\times10^{-21}\ \text{kg}\cdot\text{m/s}$$

(b) What is the approximate zero-point energy corresponding to this particle being confined in this small space? Give your answer in keV (= 1.609×10^{-16} J).

The kinetic energy of a particle is $p^2/2m$, which if we use our uncertainty, leads to an energy of

$$E = \frac{(\Delta p)^2}{2m} = \frac{\left(2.20\times10^{-21}\ \text{kg}\cdot\text{m/s}\right)^2}{2\left(1.673\times10^{-27}\ \text{kg}\right)} = 1.44\times10^{-15}\ \text{J} = 8.97\ \text{keV}$$

(c) What does the energy change to if we put a barrier half way through the box, as illustrated by the dashed line, so that the proton is forced to be on one side or the other of the barrier? How much work would be required to physically force the barrier into place?

If you put a barrier half way through the box, the position uncertainty gets cut in half, and therefore the uncertainty in the momentum doubles. This quadruples the kinetic energy, to about $E = 35.9$ keV. Where did this increased energy come from? From inserting the barrier, which must require an amount of work equal to the difference in energy, which is

$$W = \Delta E = \left(4E - E\right) = 3\left(8.97\ \text{keV}\right) = 26.9\ \text{keV}.$$

17. A block of cadmium is bombarded with photons of wavelength λ = 187 nm. It is discovered that electrons are ejected with energy 2.56 V.

(a) What is the work function ϕ for cadmium?

The initial photon has an energy of

$$E = hf = \frac{hc}{\lambda} = \frac{\left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)}{1.87\times10^{-7}\ \text{m}} = 6.63\ \text{eV}$$

The energy of the photon went into freeing the electrons, with a leftover energy of 2.56 eV, so the work function must be

$$\phi = hf - eV = \left(6.63\ \text{eV}\right) - \left(2.56\ \text{eV}\right) = 4.07\ \text{eV}$$

(b) What is the longest wavelength light that can eject an electron from cadmium?

The lowest frequency that can liberate an electron from cadmium is given by

$$f = \phi/h = \left(4.07\ \text{eV}\right)\Big/\left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right) = 9.84\times10^{14}\ \text{s}^{-1}$$

This corresponds to the longest wavelength that will work of

$$\lambda = c/f = \left(2.998\times10^{8}\ \text{m/s}\right)\Big/\left(9.84\times10^{14}\ \text{s}^{-1}\right) = 3.05\times10^{-7}\ \text{m} = 305\ \text{nm}$$

(c) For each of the following wavelengths, tell me how much energy an electron from cadmium will have after being ejected, or write "impossible" if it can't be done:

(i) λ = 282 nm **(ii) λ = 373 nm**

The second one is longer than the maximum wavelength, and therefore it is impossible for it to liberate an electron from cadmium. For the first one, the photon has an energy of

$$E = hf = \frac{hc}{\lambda} = \frac{\left(4.136\times10^{-15}\ \text{eV}\cdot\text{s}\right)\left(2.998\times10^{8}\ \text{m/s}\right)}{2.82\times10^{-7}\ \text{m}} = 4.40\ \text{eV}$$

Since 4.07 eV of this is required to remove the electron from cadmium, there is only 0.33 eV left over as the kinetic energy of the electron after ejection from cadmium.