Solutions to Test 1
September 18, 2009

Possibly useful formulas:

$$\vec{F} = q\vec{E} + q\vec{u}\times\vec{B}$$
$$p = qRB$$
$$f = \frac{f_0}{\gamma\left(1 - v\cos\theta/c\right)}$$

$$x' = \gamma\left(x - vt\right)$$
$$t' = \gamma\left(t - vx/c^2\right)$$
$$y' = y$$
$$z' = z$$

$$E' = \gamma\left(E - vp_x\right)$$
$$p'_x = \gamma\left(p_x - vE/c^2\right)$$
$$p'_y = p_y,$$
$$p'_z = p_z$$

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$
$$u'_y = \frac{u_y}{\gamma\left(1 - vu_x/c^2\right)}$$
$$u'_z = \frac{u_z}{\gamma\left(1 - vu_x/c^2\right)}$$

$$\left(1+\varepsilon\right)^n = 1 + n\varepsilon + \tfrac{1}{2}n\left(n-1\right)\varepsilon^2 + \cdots$$

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. What distinguishes “good” coordinate changes, like the Lorentz boost and rotation, from “bad” ones, like a rescaling coordinate change?
   A) They leave the (four dimensional) distance unchanged
   B) The origin of coordinates does not accelerate under good coordinate choices
   C) Angles always remain the same under good coordinate choices
   D) Good coordinate changes do not mix time with distance
   E) Good coordinate changes always involve angles of rotation

2. The Michelson-Morley experiment demonstrated that
   A) The Earth moved around the Sun (only)
   B) The Earth moved compared to the background ether (only)
   C) The Solar System orbited around the center of the galaxy (only)
   D) A, B, and C are all correct
   E) Actually, it demonstrated no motion of the Earth compared to anything.

3. Particle colliders like the Large Hadron Collider make charged particles (protons) move in a giant circle by pushing on them using
   A) Photonic fields
   B) Gravitational fields
   C) Electric fields
   D) Magnetic fields
   E) Little elves

4. Which of the following <u>is</u> a conservation law in special relativity?
   A) Energy (only)
   B) Momentum (only)
   C) Mass (only)
   D) Energy and momentum, but not mass
   E) Energy, momentum, and mass

5. The force and work formulas from non-relativistic physics are $\vec{F} = m\vec{a}$ and $W = \vec{F} \cdot \vec{d}$. Which of these are valid in special relativity?
   A) $\vec{F} = m\vec{a}$ only **B) $W = \vec{F} \cdot \vec{d}$ only** C) Both D) Neither
   E) I'm not going to do the Work on this one, and you can't Force me to.

6. According to the conventions I used in class, the mass of an object:
   A) Is equal to the sum of the masses of each part
   B) Is the same at all speeds; it is independent of speed
   C) Is equal to E/c^2
   D) Will not change if you add, say, chemical energy to the object
   E) None of these are true

7. The frequencies from a moving source are red-shifted by the greatest amount when it is moving
   A) Directly away from you
   B) Directly towards you
   C) Perpendicular to your line of sight
   D) It doesn't matter the direction, only the speed
   E) Frequencies can't be red-shifted

8. Clock A is at rest, and clock B moves past it at high velocity. Observers accompanying each clock observe the other clock, and conclude
   A) Clock B is slower than clock A
   B) Clock A is slower than clock B
   C) Each person will conclude the other clock runs slowly
   D) Each person will conclude the other clock runs quickly
   E) Each person sees the other clock running at normal speed

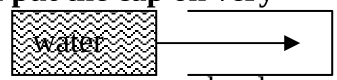
9. A cube of steel is heated from a temperature of 300 K to 600 K. As a consequence, its mass will
   A) Stay the same
   B) Decrease substantially
   C) Decrease slightly
   D) Increase substantially
   E) Increase slightly

10. Which of the following is the best estimate of γ at low velocities?
   A) $\gamma = 1 + \frac{v^2}{c^2}$ B) $\gamma = 1 - \frac{v^2}{c^2}$ **C)** $\gamma = 1 + \frac{v^2}{2c^2}$ D) $\gamma = 1 - \frac{v^2}{2c^2}$ E) $\gamma = 1$

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (1-3 sentences, or a simple sketch) (10 points each).

11. The Relativity Water Bottling company finds a new way to get money from consumers. They take more than a liter of water, accelerate it to a substantial fraction of the speed of light (basically, firing the water into a VERY sturdy water bottle), so that it is Lorentz contracted, and then put the cap on very quickly, while the water is still moving. They sell the water, advertising "more water per liter!" Would this (in principle) work? If impossible, explain why. If possible, explain how.



As viewed in our frame, the water is shorter than the size of the bottle, and therefore will fit. As viewed in the frame of the water, it does not fit. Nonetheless, as the leading edge of the water hits the back of the bottle, the trailing edge will continue moving (it can't know the front end has hit the bottle back yet), and therefore it will continue in and end up all inside. So this works, at least in principle. This isn't really surprising, since in special relativity, there are no rigid objects, so the water is guaranteed to be squishy. It just ends up in a compressed state.

12. When a rocket sends us radio signals, the frequency of the received radio waves is measured *very* carefully. Why do we do this, since presumably the radio builders can simply *tell* us what frequency it was designed to transmit?

The radio builders can tell us the frequency at which the radio is transmitting, but because the rocket is moving, this will *not* be the same as what frequency is observed, which is related to it by the Doppler shift equation,

$$f = \frac{f_0}{\gamma\left(1 - v\cos\theta/c\right)}$$

Measuring the frequency precisely can therefore give us detailed information about the velocity and/or angle it is moving.

13. Two observers are moving at velocity *v* compared to each other. Describe an experiment that would determine which of them is actually moving, or argue that no such experiment is possible.

According to relativity, there is no such thing as absolute velocity. Each observer has an equally valid reference frame, the laws of physics are identical, and no physical experiment can determine which one is "actually" moving.

Part III: Calculation: [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each)

14. Due to a low battery in my watch, I discover that it is advancing slowly, so it advances only 50 seconds in every minute. No problem! I will simply run around at very high velocity *v* so that as viewed by me, all other clocks run slowly as well.

(a) How fast do I need to run in order to make other clocks appear to run at the same pace as mine?

Other clocks will run more slowly by a factor of γ. We want them to run at $\frac{5}{6}$ of the speed, so $\gamma = \frac{6}{5} = 1.2$. We therefore have

$$1.2 = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

$$1 - v^2/c^2 = \frac{1}{1.2^2} = 0.6944$$

$$v^2/c^2 = 1 - 0.6944 = 0.3056$$

$$v/c = \sqrt{0.3056} = 0.5528, \quad \text{so} \quad v = 0.5528c = 1.656 \times 10^8 \text{ m/s}$$

(b) According to other observers, how many seconds pass on *my* watch in a minute. Keep in mind that my watch runs slow!

Those observers will see time run more slowly for me, by the same factor γ. According to them, in 60 seconds of their time, only 50 seconds of my time will pass. But since my watch is running slow, it will only advance $\frac{5}{6}$ of this amount, or about 41.7 seconds.

(c) Arriving at class one minute early, I decide I might as well run around campus at this high speed and get a few errands done. I arrive back when my second hand has advanced 60 seconds. Will I return early, late, or exactly on time?

Because the room remains at constant speed, while I am accelerating all over the place to get my errands done, the room will actually be older, or in fact, my watch will actually turn out to have run more slowly. Since it takes more than 60 seconds of my time for my watch to advance that amount, then as experienced by me, *more than* 60 seconds will have elapsed, and according to the wall clock in the room, even more time than that will have elapsed. I will return late; in fact, one can show I will return a full 26.4 seconds late.

15. Three events all have coordinates $y = z = 0$, but they have different x and t coordinates, as given in the table at right. For each part of the problem, assume the numbers given at right are exact, and the speed of light is exactly $c = 3\times10^8$ m/s

(a) What is the proper distance (in m) or proper time (in ns) separating *A* from *B*? Is one of these events in the absolute future of the other (which one?), is one on the future light cone of the other, or are they "elsewhere" from one another?

Point	x	t
A	0 m	0 ns
B	5 m	10 ns
C	6 m	20 ns

For each part of this, we need to calculate the proper distance $s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$. When it comes out positive, we take the square root and call it *s*. When it comes out negative, we set it equal to $-c^2\tau^2$, so we take the negative, take the square root, and divide by *c* to get the proper time. If it comes out zero, the separation is lightlike, and we either don't bother taking the square root or write $s = 0$, if we prefer. For this part we have

$$s^2 = (\Delta x)^2 - (c\Delta t)^2 = (5\text{ m})^2 - \left[\left(3\times10^8\text{ m/s}\right)\left(10\times10^{-9}\text{ s}\right)\right]^2 = 25\text{ m}^2 - 9\text{ m}^2 = 16\text{ m}^2,$$

$$s = 4\text{ m}.$$

Since the separation came out spacelike, the pair of points are elsewhere from each other.

(b) Repeat, this time comparing *A* to *C*.

As before, we find

$$s^2 = (\Delta x)^2 - (c\Delta t)^2 = (6\text{ m})^2 - \left[\left(3\times10^8\text{ m/s}\right)\left(20\times10^{-9}\text{ s}\right)\right]^2 = 36\text{ m}^2 - 36\text{ m}^2 = 0.$$

We can write $s = 0$, if we wish. Since the separation is lightlike, they must be on each other's light cones. Obviously, *C* is on the future light cone of *A* (and *A* is on the past light cone of *C*).

(c) Repeat, this time comparing *B* to *C*.

$$s^2 = (\Delta x)^2 - (c\Delta t)^2 = (1\text{ m})^2 - \left[\left(3\times10^8\text{ m/s}\right)\left(10\times10^{-9}\text{ s}\right)\right]^2 = 1\text{ m}^2 - 9\text{ m}^2 = -8\text{ m}^2.$$

Since we got a negative number, we set it equal to $-c^2\tau^2$, so that

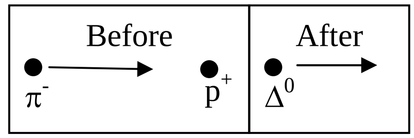
$$c^2\tau^2 = 8\text{ m}^2,$$

$$c\tau = \sqrt{8}\text{ m} = 2.828\text{ m},$$

$$\tau = \frac{2.828\text{ m}}{3.00\times10^8\text{ m/s}} = 9.427\times10^{-9}\text{ s} = 9.427\text{ ns}.$$

Since it came out timelike, one is in the future of the other. Obviously, it is *C* that is in the future of *B* (and *B* is in the past of *C*).

16. **A pion (π^-) of mass m_π = 140 MeV/c^2 is moving with velocity v = 0.905c in the +x direction. It collides with a stationary proton (p^+) of mass m_p = 938 MeV/c^2. The two particles merge to form a Δ^0 particle.**



(a) What is the energy (in MeV) and momentum (in MeV/c) of the pion? How about the proton?

For the pion, we need γ, which is given by

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.905^2}} = 2.351$$

We can then get the energy and momentum in a straightforward manner.

$$E_\pi = \gamma m_\pi c^2 = (2.351)\left(140 \text{ MeV}/c^2\right)c^2 = 329 \text{ MeV}$$
$$p_\pi = \gamma m_\pi v = (2.351)\left(140 \text{ MeV}/c^2\right)(0.905c) = 298 \text{ MeV}/c$$

The proton is at rest, and therefore has no momentum, and an energy of $E_0 = m_p c^2$, so

$$E_p = 938 \text{ MeV}$$
$$p_p = 0$$

(b) What is the energy and momentum of the final Δ^0, in the same units?

By conservation of energy and momentum, the Delta must have energy and momentum of

$$E_\Delta = E_p + E_\pi = 938 \text{ MeV} + 329 \text{ MeV} = 1267 \text{ MeV}$$
$$p_\Delta = p_p + p_\pi = 0 + 298 \text{ MeV}/c = 298 \text{ MeV}/c$$

(c) What is the mass of the Δ^0 in MeV/c^2? What is its velocity as a fraction of c?

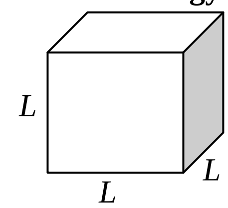
We can use the mass formula to tell us

$$m_\Delta^2 c^4 = E_\Delta^2 - c^2 p_\Delta^2 = (1267 \text{ MeV})^2 - (298 \text{ MeV})^2 = 1.516 \times 10^6 \text{ MeV}^2$$
$$m_\Delta c^2 = 1231 \text{ MeV}$$

Then we use the velocity formula to tell us

$$\frac{u}{c} = \frac{pc}{E} = \frac{298 \text{ MeV}}{1267 \text{ MeV}} = 0.235$$

17. A cube of dimensions *L* × *L* × *L* is at rest. It is filled with a material with energy density ρ (energy per unit volume) and momentum density 0 (momentum per unit volume). The box is then viewed by an observer moving at speed *v* = 0.6*c* in the *x*-direction.



(a) What are the dimensions of the box as viewed by the moving observer?

Such a box moving at high speed would be Lorentz contracted by a factor of γ at that speed, where

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0.600^2}} = 1.250, \quad \gamma^{-1} = 0.800$$

Therefore, the dimensions of the box will now be $L \times L \times 0.8L$.

(b) What are the energy and the momentum of the box in its rest frame? What is the energy and momentum of the box as viewed by the moving observer?

The energy of the original box is just the energy density times the volume, or $E = \rho L^3$. Since the momentum density is zero, the momentum is $p = 0$. For the moving box, we simply use the formula for Lorentz boost of energy and momentum:

$$E' = \gamma\left(E - vp_x\right) = (1.25)\left(\rho L^3\right) = 1.25\rho L^3,$$
$$p'_x = \gamma\left(p_x - vE/c^2\right) = -(1.25)\cdot(0.6c)\left(\rho L^3\right)/c^2 = -0.75\rho L^3/c,$$
$$p'_y = p_y = 0,$$
$$p'_z = p_z = 0.$$

(c) What is the energy density and momentum density of the box as viewed by the moving observer?

To find these, simply divide the answers from the previous part by the volume found in part (a), which yields

$$\rho' = E'/V' = \left(1.25\rho L^3\right)/\left(0.8L^3\right) = 1.5625\rho,$$
$$j'_x = p'_x/V' = \left(-0.75\rho L^3/c\right)/\left(0.8L^3\right) = -0.9375\rho/c,$$
$$j'_y = j'_z = 0.$$