Name ___________________

Solutions to Test 1
September 21, 2007

This test consists of three parts. Please note that in parts II and III, you can skip one question of those offered.

Possibly useful formulas:

$$f = \frac{f_0}{\gamma\left(1 - v\cos\theta/c\right)}$$

$$p'_x = \gamma\left(p_x - vE/c^2\right)$$
$$E' = \gamma\left(E - vp_x\right)$$
$$p'_y = p_y$$
$$p'_z = p_z$$

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$
$$u'_y = \frac{u_y}{\gamma\left(1 - vu_x/c^2\right)}$$
$$u'_z = \frac{u_z}{\gamma\left(1 - vu_x/c^2\right)}$$

$$p = qRB$$

$$\left(1+\varepsilon\right)^n = 1 + n\varepsilon + \tfrac{1}{2}n\left(n-1\right)\varepsilon^2 + \cdots$$

$$\vec{F} = q\vec{\mathrm{E}} + q\vec{u}\times\vec{\mathrm{B}}$$

Part I: Multiple Choice [20 points]
For each question, choose the best answer (2 points each)

1. Two incoming particles collide which are moving in the x-direction. Two particles come out afterwards, now moving in an arbitrary direction. Which of the following will *not* generally be conserved?
   A) Mass
   B) Energy
   C) Momentum in the x-direction
   D) Momentum in the y-direction and the z-direction
   E) Actually, all of these are conserved

2. What is meant by the term *world line* (misspelled *worldline* in the lectures) in special relativity?
   A) A line representing a set of points all at the same time, like the x-axis
   B) A line representing a set of points all at the same place, like the t-axis
   C) A line representing any axis of some observer moving compared to us
   D) A line tilted at a 45-degree angle on a space-time diagram
   E) The path of some material object through space and time

3. The Lorentz boost formulas for coordinates (called by the book Lorentz transformations) tell you how to change coordinates when
   A) Comparing two observers that are at an angle compared to each other in space
   B) Comparing two observers that are at the same speed and angle, but one has shifted their space origin of coordinates
   C) Comparing two observers that are at the same speed and angle, but one has shifted their time origin of coordinates
   D) Comparing two observers that are oriented in opposite directions compared to time
   E) Comparing two observers that are moving compared to each other

4. When an object is moving very close to the speed of light, its linear dimensions, as viewed by us, will
   A) Shrink in the direction of motion, but not in perpendicular directions
   B) Expand in the direction of motion, but not in perpendicular directions
   C) Shrink in perpendicular directions, but not in the direction of motion
   D) Expand in perpendicular directions, but not in the direction of motion
   E) Neither shrink nor expand in any dimension as viewed by us

5. In the Large Hadron Collider, currently under construction in Geneva, protons are forced to go around a very large circular track. The method used to keep them going in circles is probably
   A) Collisions with the walls of the collider
   B) Gravitational force
   C) Electric fields
   D) Magnetic fields
   E) Nuclear fields

6. In the classic version of the twin paradox, where one observer goes to a star and returns, while the other stays here, the reason there is an absolute difference between the two observers is that
   A) One observer was really moving, while the other wasn't
   B) One observer was moving compared to other objects (like the Sun), while the other wasn't
   C) One observer always moved at constant velocity, while the other didn't
   D) One observer was always near a gravitational source, while the other wasn't
   E) One observer moved only in the time direction, while the other also moved in the space direction

7. Under what condition(s) is it perhaps advisable to expand the Lorentz factor γ or its inverse using the binomial expansion?
   A) When the object is moving nearly at the speed of light
   B) When the object is moving much slower than the speed of light
   C) When calculating time dilation, but not Lorentz contraction
   D) When calculating Lorentz contraction, but not time dilation
   E) When calculating Lorentz contraction or time dilation, but not Lorentz transformations

8. Suppose we mix two chemicals, and they react in such a way that energy is released (which then escapes from the mixture). How will the total mass of the final combination be different from the sum of the masses of the constituents?
   A) It will be a lot smaller
   B) It will be very slightly smaller
   C) It will be exactly the same
   D) It will be very slightly larger
   E) It will be a lot larger

9. Why does energy appear to get "mixed up" with momentum in the formulas above for Lorentz transformations of energy and momentum?
   A) Changing speeds involves emitting light rays, which can convert energy into momentum and vice versa
   B) As you change speed, you add energy, and therefore you change the mass which changes the momentum
   C) Energy and momentum are different parts of the same four-dimensional vector, and therefore naturally mix together when you perform a Lorentz boost
   D) Any method for changing speeds involves electric and magnetic fields, which actually perform this change for you
   E) I have no idea; please mark this one wrong

10. As an object of finite mass *m* approaches the speed of light *c*, what happens to the object's momentum and energy?
   A) They both go to large but finite values
   B) Momentum goes to infinity, but energy remains finite
   C) Energy goes to infinity, but momentum remains finite
   D) They both go to infinity
   E) There is insufficient information to answer this question

Part II: Short answer [20 points]

Choose **two** of the following questions and give a short answer (1-3 sentences) (10 points each).

11. Dr. Carlson has just devised an instantaneous communication device! It consists of a 1 light-year long steel rod, together with two people who know Morse code. One person pushes/pulls the rod at one end to send the message, and the other one watches her end to see what happens. Tell me what you think of Dr. Carlson's invention.

Dr. Carlson's invention assumes that when you move one end of the rod, the other and moves simultaneously. However, the term *simultaneous* is ambiguous in relativity, and furthermore, relativity states that you can never transmit influences faster than the speed of light. In fact, since there are no rigid objects in relativity, when you move one end of the rod, the other end reacts not instantaneously, and not even at the speed of light, but merely at the speed of sound in steel, which is much less than the speed of light.

12. Qualitatively, how does the frequency of absorbed radiation from a distant source change if it is moving (a) towards you, (b) away from you, or (c) perpendicular to you. You don't have to use formulas, but you have to tell me if it increases or decreases or stays the same, and perhaps something about the size of the effect.

When an object is moving towards you, the frequency is "blue shifted" towards higher frequency, and when it is moving away, it is "red shifted" towards lower frequency. When it is moving perpendicular to you, it is also shifted towards lower frequencies, but the effect (for small velocities) is significantly less than it is for when it is moving away from you.

13. What makes transformations like rotations and Lorentz boosts good, while transformations like the "skew" transformation (that distorted my picture) and the Galilean boost bad, according to special relativity? Use of an equation in your answer is strongly encouraged.

Good transformations are those that preserve the distance formula; bad ones are ones that do not. The distance formula is given, in four dimensions, by

$$s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2 (\Delta t)^2$$

Rotations and Lorentz boosts leave this formula unchanged; skew transformations and Galilean boosts do not.

Part III: Calculation: [60 points]

Choose **three** of the following four questions and perform the indicated calculations (20 points each)

14. An observer is currently at the point in space-time $(x_1, y_1, z_1, t_1) = (2 \text{ m}, 3 \text{ m}, 4 \text{ m}, 0 \text{ s})$. A second observer is stationary at $(x_2, y_2, z_2, t_2) = (0 \text{ m}, 2 \text{ m}, 6 \text{ m}, t)$ where *t* is unknown.

(a) For what values of *t* will the separation between these two points be timelike, spacelike, or lightlike? You may give your answer either in terms of *t* or *ct*, whichever you prefer.

(b) For which values of *t* will the second point be in the absolute past, absolute future, past light cone, future light cone, or elsewhere, compared to the first point?

It is actually easier to do both parts together, so that's how we'll proceed. We first need to find the proper distance squared, or s^2, which is given by

$$\begin{aligned} s^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \\ &= (0 \text{ m} - 2 \text{ m})^2 + (2 \text{ m} - 3 \text{ m})^2 + (6 \text{ m} - 4 \text{ m})^2 - c^2 (t - 0)^2 \\ &= 9 \text{ m}^2 - c^2 t^2 \end{aligned}$$

When this is positive, the separation is spacelike; that is we have spacelike separation if

$$\begin{aligned} 9 \text{ m}^2 &> c^2 t^2 \\ |ct| &< 3 \text{ m} \end{aligned}$$

In this case, the points are designated as "elsewhere", and therefore it is meaningless to figure out if they are in the future or the past. When this expression vanishes, then we have a lightlike separation. This clearly occurs when $|ct| = 3 \text{ m}$. This has two solutions, the positive one corresponding to the future and the negative one corresponding to the past. Finally, it is easy to see that the expression for s^2 will be negative when $|ct| > 3 \text{ m}$. This breaks into two solutions, one with $ct < -3$ m, which corresponds to the absolute past, and $ct > 3$ m, which corresponds to the absolute future. We can divide the quantity *ct* by *c* if we want, in which case we find that the break points come at $t = \pm 10 \text{ ns}$. In summary, all times can be divided into five categories as indicated in the table below.

Category	Separation	*ct*	*t*
Absolute Future	timelike	$ct > 3$ m	$t > 10$ ns
Future Light Cone	lightlike	$ct = 3$ m	$t = 10$ ns
Elsewhere	spacelike	$-3 \text{ m} < ct < 3$ m	$-10 \text{ ns} < t < 10$ ns
Past Light Cone	lightlike	$ct = -3$ m	$t = -10$ ns
Absolute Past	timelike	$ct < -3$ m	$t < -10$ ns

15. A 1.50 m tall man plus his rocket ship (total mass 100 kg) are to be accelerated from rest to 2.13 × 10⁸ m/s. This is to be achieved by a constant force of 980 N (about the weight on Earth).

(a) What is the initial energy and momentum of the system, in J and kg·m/s? What is the final momentum and final energy of the system, in the same units?

The initial velocity is zero, so the initial energy and momentum are given by

$$p_i = \gamma m u = 0$$

$$E_i = \gamma mc^2 = mc^2 = (100\text{ kg})\left(3.00\times10^8\text{ m/s}\right)^2 = 8.99\times10^{18}\text{ J}$$

The final velocity gives

$$p_f = \gamma m u = \frac{(100\text{ kg})\left(2.13\times10^8\text{ m/s}\right)}{\sqrt{1-(2.13/3.00)^2}} = 3.03\times10^{10}\text{ kg}\cdot\text{m/s}$$

$$E_i = \gamma mc^2 = \frac{(100\text{ kg})\left(3.00\times10^8\text{ m/s}\right)^2}{\sqrt{1-(2.13/3.00)^2}} = 12.77\times10^{18}\text{ J}$$

(b) How long does it take to reach this speed, as measured by us? Give the answer in years (1 yr = 3.156×10^7 s).

We will use the formula $F = dp/dt$. Since the force is constant, this can be easily integrated to yield $\Delta p = Ft$. Solving for the time t, we have

$$t = \frac{\Delta p}{F} = \frac{3.03\times10^{10}\text{ kg}\cdot\text{m/s}}{980\text{ kg}\cdot\text{m/s}^2} = 3.09\times10^7\text{ s}\left(\frac{1\text{ yr}}{3.156\times10^7\text{ s}}\right) = 0.979\text{ yr}$$

(c) How far has the system traveled by this time? Give the answer in light-years (1 ly = 9.461×10^{15} m).

We use the work formule, $W = Fd$, where W is the change in energy, so we have

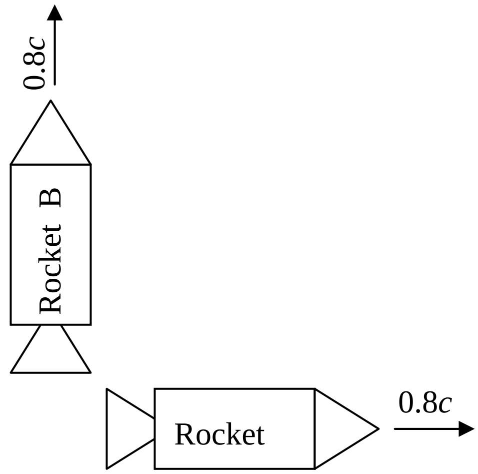
$$D = \frac{\Delta E}{F} = \frac{(12.77-8.99)\times10^{18}\text{ N}\cdot\text{m}}{980\text{ N}} = 3.86\times10^{15}\text{ m}\left(\frac{1\text{ ly}}{9.461\times10^{15}\text{ m}}\right) = 0.408\text{ ly}$$

(d) Assuming the rocket is moving in the direction the man is standing, how tall does he seem to us?

The proper length of the man is 1.5 m, so the observed length is

$$L = L_P/\gamma = (1.5\text{ m})\sqrt{1-(2.13/3.00)^2} = 1.0?\text{ m}$$

16. Rocket A and Rocket B start at the same place at the same time, but simultaneously accelerate to 0.8*c*, but with rocket A moving in the +*x* direction and rocket B moving in the +*y* direction.



(a) Write all three components $\vec{u}$ of the velocity of rocket B in the original system (this is trivial)? Write all three components $\vec{u}'$ of the velocity of rocket B as viewed by Rocket A. I recommend writing all velocities as fractions of *c*.

The three components are $\vec{u} = (0, 0.8c, 0)$ in the original system. We use the subtraction of velocity formula to find

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} = \frac{0 - 0.8c}{1 - 0} = -0.8c,$$

$$u'_y = \frac{u_y}{\gamma\left(1 - vu_x/c^2\right)} = \frac{0.8c\sqrt{1 - 0.8^2}}{(1 - 0)} = 0.48c,$$

$$u'_z = \frac{u_z}{\gamma\left(1 - vu_x/c^2\right)} = \frac{0\sqrt{1 - 0.8^2}}{(1 - 0)} = 0.$$

So the velocity is $\vec{u}' = (-0.8c, 0.48c, 0)$

(b) Find the magnitude of the velocity $\vec{u}'^2$ of rocket B and check that it is less than c^2.

The magnitude is

$$\vec{u}'^2 = 0.8^2 c^2 + 0.48^2 c^2 = (0.64 + 0.2304)c^2 = 0.8704c^2$$

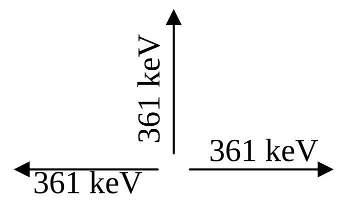
which is, of course, less than c^2.

(c) Unfortunately for the person on Rocket B, after 2.73 hours of travel (as measured by B), rocket B develops engine trouble. According to the observer on rocket A, how much time has passed when this occurred?

The proper time is the amount given, so we use the formula $\Delta t = \gamma\tau$ to find the proper time, where the Lorentz factor is based on the velocity found in part (b), so we have

$$\Delta t = \gamma\tau = \frac{\tau}{\sqrt{1 - \vec{u}'^2/c^2}} = \frac{2.73\text{ hr}}{\sqrt{1 - 0.8704}} = 7.58\text{ hr}$$

17. An Ortho-Positronium particle that is <u>not</u> at rest decays to three photons. Suppose that the three photons all have mass 0, energy $E_{1,2,3}$ = 361.3 keV, and are moving in the $+x$, $-x$, and $+y$ direction respectively, as illustrated at right.



(a) What is the momentum of each of the three photons?

The photons are massless, and therefore satisfy $E = pc$. Therefore, each of them has momentum $p = E/c$ = 361.3 keV/c. Of course, momentum is a vector, so each of these has a direction as well, so we have $\vec{p}_1 = 361.3\hat{i}\ \text{keV}/c$, $\vec{p}_2 = -361.3\hat{i}\ \text{keV}/c$, and $\vec{p}_3 = 361.3\hat{j}\ \text{keV}/c$.

(b) What is the initial energy in keV and momentum in keV/c of the ortho-positronium?

We simply use conservation of energy and momentum.

$$E = E_1 + E_2 + E_3 = 3(361.3)\ \text{keV} = 1083.9\ \text{keV},$$
$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 361.3(\hat{i} - \hat{i} + \hat{j})\ \text{keV}/c = 361.3\hat{j}\ \text{keV}/c$$

(c) What is the mass of ortho-positronium, in keV/c^2?

We can calculate this most efficiently using the mass formula

$$(mc^2)^2 = E^2 - (\vec{p}c)^2 = (1083.9\ \text{keV})^2 - (361.3\hat{j}\ \text{keV})^2 = 1.044\times10^6\ \text{keV}^2,$$
$$mc^2 = 1022\ \text{keV},$$
$$m = 1022\ \text{keV}/c^2.$$