III. Particle Physics and Isospin

Up to now we have concentrated exclusively on actual, physical rotations, either of coordinates or of spin states, or both. In this chapter we will be concentrating on internal symmetries, symmetries or approximate symmetries that have nothing to do with physical rotations. These symmetries are useful in a variety of settings, but probably have the widest application in particle physics.

A. Particle Physics

Particle physics, also known as quantum field theory, comes from the union of Einstein's special theory of relativity and quantum mechanics. According to relativity, information can never be transmitted faster than the speed of light, and the usual instantaneous action at a distance we commonly posit in conventional quantum mechanics is violated. All forces and interactions must be transmitted by fields, similar to the familiar electromagnetic fields, with which you are already familiar. Indeed, it is not just the *forces* that are described in terms of fields, but everything, including electrons, protons, etc. Hence the name quantum field theory.

When you perform a rotation of a field, it will change in some predictable way. For example, when you rotate an electric or a magnetic field, it will rotate as a vector, with the components changing into each other in a predictable way. How the fields rotate ends up influencing the properties of the resulting quantized particles. Hence the resulting particle (the photon) will have a predictable spin (1 in the case of the photon).[1] Similarly, all electrons are spin ½. The possible spins of a particle are, of course, simply integers or half-integers, the same as in ordinary quantum physics. It is a prediction of quantum field theory that any half-integer spin field will result in particles which must be in anti-symmetric quantum states (fermions) while integer spin fields result in particles which must be in symmetric quantum states (bosons).

As you may be familiar with from quantizing the electromagnetic field, when you quantize a field you end up with a formalism sufficiently powerful to deal with the creation and destruction of particles. There is no conservation-of-particles rule in particle physics. The field theory must include information (in the Hamiltonian) that tells you how much energy you need to create a particle, which for a particle at rest is simply $E = mc^2$. Hence the Hamiltonian must contain information about the mass of the particles as well.[2] For technical reasons, it turns out that the term in the Hamiltonian is proportional to m^2 for bosons and to m for fermions. I'm not going to try to explain why.

In ordinary quantum mechanics, we perform rotations in 3D space; in quantum field theory we also perform Lorentz transformations. The full Lorentz group is interesting and not terribly difficult (it has the name $SO(3,1)$, and is closely related to $SO(4)$), but it isn't compact (basically because no matter how much you accelerate, you

[1] It's a bit more complicated than this. It turns out the field you end up quantizing is neither the electric nor the magnetic field, but the vector potential $\mathbf{A}(\mathbf{r})$. Any way, it works out to spin 1.

[2] In this class, the word *mass* always refers to rest mass. Hence all electrons have the same mass, regardless of their velocity.

never “circle around” back to where you started), and we won’t discuss it here. A closely related phenomenon is the existence of anti-particles. Every field, it turns out, creates and annihilates *two* types of particles, one of which we call the particle and the other the anti-particle (which is which is a matter of convention). The anti-particle always has the same spin and the same mass as the particle. Occasionally they will be identical. It is *also* the case that if we have a symmetry group $\mathcal{G}$ of a set of particles that transform under a representation Γ, their corresponding anti-particles will transform under the representation Γ^*. It follows that if a particle is its own anti-particle, it must only transform under real representations of the group. I know this was confusing, but we’ll return to this in the next section.

In addition to the symmetries of the Lorentz group, it is not uncommon to have so-called internal symmetries. These symmetries describe changes that interchange one particle for another, rotating them in some abstract space, rather than physical space. These symmetries can be either discrete or continuous, but the continuous ones seem to be used more often, and can also correspond to forces, as we will discuss in the next section.

B. *U*(1) Symmetries

I want to start the discussion of symmetries with a familiar example. Suppose we have a single electron with wave function $\left|\psi(\mathbf{r})\right\rangle$. It is well known that if $\left|\psi(\mathbf{r})\right\rangle$ is a solution of Schrödinger’s equation, so is $e^{i\theta}\left|\psi(\mathbf{r})\right\rangle$. One way to think of this is that there is an operator $R(\theta)$ which has the effect

$$R(\theta)\left|\psi(\mathbf{r})\right\rangle = e^{i\theta}\left|\psi(\mathbf{r})\right\rangle \tag{3.1}$$

Since the set of numbers $e^{i\theta}$ can be thought of as unitary 1×1 matrices, this is the group *U*(1). According to chapter 1, the representation $\Gamma^{(q)}$ is given by $\Gamma^{(q)} = e^{-iq\theta}$, so we see that the electron corresponds to $q=-1$. If we call this the “charge” then we would say the electron has charge -1.

What if we have two electrons? Well, the wave function would look something like $\left|\psi(\mathbf{r}_1,\mathbf{r}_2)\right\rangle$, which is some component of the tensor product space of the two separate wave functions $\left|\psi_1(\mathbf{r}_1)\right\rangle \otimes \left|\psi_2(\mathbf{r}_2)\right\rangle$, and therefore we find

$$R(\theta)\left|\psi(\mathbf{r}_1,\mathbf{r}_2)\right\rangle = e^{2i\theta}\left|\psi(\mathbf{r}_1,\mathbf{r}_2)\right\rangle \tag{3.2}$$

This leads us to the remarkable conclusion that two electrons have a total charge of -2. I will leave it to the reader to deduce the charge of three or more electrons.

Other particles will have other charges; for example, the proton has charge +1, the Δ^{++} charge +2, the uranium nucleus +92. What about anti-particles? The representation under the group *U*(1) of the positron (anti-electron) should correspond to the complex conjugate of the “matrices” appearing in (3.1). Therefore, for a positron, we would have

$$R(\theta)\left|\psi(\mathbf{r})\right\rangle = e^{-i\theta}\left|\psi(\mathbf{r})\right\rangle \tag{3.3}$$

Hence positrons are predicted to have charge +1. This is a general rule for a *U*(1) symmetry: the "charge" of the anti-particle is the opposite of the "charge" of the particle.

It is useful to work with the operator that is the generator of this abstract phase rotation, which we will (in this case) call *Q*, defined by

$$Q = i\frac{\partial}{\partial\theta}R(\theta)\Big|_{\theta=0} \tag{3.4}$$

where *Q* is called the charge operator. It has the value -1 on the electron, +2 on the Δ^{++}, etc. We can then easily show

$$R(\theta) = \exp(-iQ\theta) \tag{3.5}$$

In the first part of this course, we commonly used symmetry to demonstrate degeneracy. This unfortunately doesn't help us in this case, since all the irreps of *U*(1) are one-dimensional, and therefore we only discover that each particle has the same mass as itself.

Now, we want *U*(1) to be a symmetry of the Hamiltonian. It follows that $\left[R(\theta), H\right] = 0$, which tells us in turn that $\left[Q, H\right] = 0$. Now, suppose we have an initial collection of particles $\left|\psi_I\right\rangle$ and a final collection $\left|\psi_F\right\rangle$. There will be some total charge associated with each of these, so

$$Q\left|\psi_I\right\rangle = q_I\left|\psi_I\right\rangle \quad \text{and} \quad Q\left|\psi_F\right\rangle = q_F\left|\psi_F\right\rangle \tag{3.6}$$

If one is going to get turned into the other, there must be some (direct or indirect) connection between them. Roughly speaking, we need $\left\langle\psi_F\right|H\left|\psi_I\right\rangle \neq 0$. We now note that

$$0 = \left\langle\psi_F\right|\left[Q, H\right]\left|\psi_I\right\rangle = \left\langle\psi_F\right|QH\left|\psi_I\right\rangle - \left\langle\psi_F\right|HQ\left|\psi_I\right\rangle = \left(q_F - q_I\right)\left\langle\psi_F\right|H\left|\psi_I\right\rangle \tag{3.7}$$

Thus the matrix element must vanish unless $q_I = q_F$. We now come to our first group theory conclusion (drum roll please): charge is conserved!

Electric charge is *not* the only *U*(1) symmetry of particle physics. Another example is so-called *baryon number*. Certain particles, such as the proton and neutron, are baryons, but so are a host of less familiar particles, such as the Δ^{++} mentioned earlier. The operator that measures baryon number is called *B* (the analog of *Q*) and the proton and neutron each have baryon number +1. Anti-protons and anti-neutrons each have baryon number -1. Electrons have baryon number 0. A ^{238}U atom (electrons and all) has baryon number +238. As far as we can tell, baryon number is absolutely conserved, and commutes with the Hamiltonian as well. The conservation of baryon number accounts for the stability of the proton and the stability or meta-stability of various isotopes: there is no lighter baryon than the proton that the proton could decay into.

There is a big difference between these two *U*(1) symmetries. Technically, electromagnetism is a *local* or *gauged* *U*(1) symmetry. This means that you can actually multiply the wave function by different phases at different points in space and time. How this is achieved is discussed in quantum mechanics course, and will not be discussed here. However, three properties that result in this case are worth mentioning:

- There is a field (the EM field), and hence a particle (the photon), with spin 1 associated with this symmetry.
- The strength of the corresponding coupling is proportional to the charge q. In other words, high charge particles make bigger EM fields.
- The most stable (lowest energy) states tend to correspond to $q = 0$.

Note that the charge q is the 1×1 generator of the corresponding irrep. These properties will generalize in other cases. For more complicated symmetries, the particle will still be spin 1, but there will be one such particle for each generator, the couplings will be proportional to the matrix T_a for each generator, and the most stable states will tend to be those where all the generators vanish; *i.e.*, the trivial irrep of the group.

Not all symmetries are gauged and have forces that carry them. Electromagnetism is (of course), but as far as we know, baryon number is not. There is no baryo-photon, no baryo-force, and nature doesn't prefer baryon-zero combinations.

C. Isotopes and Isospin

Our examples before were rather disappointing, because we found no degeneracies with the group $U(1)$. Let's try to find a more interesting case. Consider, for the moment, the proton and neutron, two baryons with mass differing by only about 0.14%. Is it possible that there is an approximate symmetry relating these two particles? There are other striking similarities when you look at more complicated nuclei. For example, ^{3}He and ^{3}H differ in mass by only 0.02%. They also have very similar energies for their excited states. Similar patterns can be found for heavier isotopes, for example ^{14}C, ^{14}N, and ^{14}O. This suggested to Werner Heisenberg in 1932 that there might be a symmetry relating protons and neutrons. Because of its similarity to spin, it was named isotopic spin by Eugene Wigner, but is now known as isospin.

Imagine for the moment an operator, which we will name $2\mathcal{I}_1$ (I'll explain the 2 later), which interchanges protons and neutrons, like this

$$2\mathcal{I}_1 |p\rangle = |n\rangle \quad \text{and} \quad 2\mathcal{I}_1 |n\rangle = |p\rangle \tag{3.8}$$

Of course, this operator can't *perfectly* commute with the Hamiltonian, since if it did, the proton and neutron would be exactly degenerate, and would have identical interactions (but only one of them has a charge).

Now, when we are looking at protons and neutrons, we already know that there are two other operators that are conserved, charge and baryon number. It follows that any linear combination of these will also be conserved. Consider in particular the operator $\mathcal{I}_3 = Q - \frac{1}{2}B$, which acting on a proton or neutron has the effect

$$\mathcal{I}_3 |p\rangle = \tfrac{1}{2}|p\rangle \quad \text{and} \quad \mathcal{I}_3 |n\rangle = \tfrac{1}{2}|n\rangle \tag{3.9}$$

This operator must commute exactly with the Hamiltonian.

If any pair of operators that commutes with the Hamiltonian, their commutator must also commute with the Hamiltonian. Let's define $\left[\mathcal{I}_3,\mathcal{I}_1\right]=i\mathcal{I}_2$. Then it isn't hard to work out the effects of $\mathcal{I}_2$ on the proton and neutron. We find

$$\mathcal{I}_2\left|p\right\rangle=\tfrac{1}{2}i\left|n\right\rangle \quad \text{and} \quad \mathcal{I}_2\left|n\right\rangle=-\tfrac{1}{2}i\left|p\right\rangle \tag{3.10}$$

Now, let's change our notation slightly. Let's consider the proton and neutron as two partner particles, which we will call nucleons, and relabel them as

$$\left|N_1\right\rangle=\left|p\right\rangle \quad \text{and} \quad \left|N_2\right\rangle=\left|n\right\rangle \tag{3.11}$$

(they might also be labeled by their charges, $\left|N^+\right\rangle=\left|p\right\rangle,\left|N^0\right\rangle=\left|n\right\rangle$). Then the operation of the three operators $\mathcal{I}_a$ on these states will be described by

$$\mathcal{I}_a\left|N_i\right\rangle=\sum_j\left|N_j\right\rangle\left(I_a\right)_{ji} \tag{3.12}$$

where our three matrices I_a are given by

$$I_1=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad I_2=\frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad I_3=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{3.13}$$

We recognize (3.13) as nothing more than the three generators of the spin-1/2 representation of $SU(2)$, so $I_a=T_a^{\left(\frac{1}{2}\right)}$. The isospin of combinations of protons and neutrons can be worked out by taking tensor products of the $\left(\frac{1}{2}\right)$ representation. The resulting total isospin is normally denoted by I (instead of j), and satisfies

$$\sum_a I_a^2=I^2+I \tag{3.14}$$

When actually *working* with isospin, it is more useful to work with the operators $\mathcal{I}_\pm=\mathcal{I}_1\pm i\mathcal{I}_2$. When acting on a proton or neutron, these have the effect

$$\mathcal{I}_+\left|n\right\rangle=\left|p\right\rangle,\quad \mathcal{I}_+\left|p\right\rangle=0,\quad \mathcal{I}_-\left|p\right\rangle=\left|n\right\rangle,\quad \mathcal{I}_-\left|n\right\rangle=0 \tag{3.15}$$

Note that $\mathcal{I}_+$ always increases the charge by one, while $\mathcal{I}_-$ always decreases it by one. This property will generalize to nuclei made of protons and neutrons, and later to other particles as well.

Since isospin pretty successfully describes the masses and energy levels of isotopes, it clearly is also saying something about their binding interactions. Isospin is not a good symmetry of electromagnetism (it treats the proton and neutron as the same!) and in fact EM interactions are probably the largest source of isospin violation. The nucleus must be bound by some other type of force, called the *strong force* (or sometimes, the strong nuclear force). Hence isospin is considered an approximate symmetry only of the *strong* force. It is not used to describe particles that do not participate in the strong force, like electrons. But the nucleons are not the *only* particles that feel the strong force. We now turn our attention to some of these other particles.

D. Other strongly interacting particles

I have in my office a copy of the particle data book, containing hundreds of discovered particles, and save for nine of them,[3] they all have strong interactions. The strongly interacting particles are categorized into three categories, the baryons, which have baryon number +1, the anti-baryons, which have baryon number -1, and the mesons, which have baryon number 0. The lightest baryons are listed at right, and the lightest mesons below. The anti-baryons are the anti-particles of the baryons, and they are not normally listed separately. The mesons are sometimes their own anti-particles, and

Baryons (mass $\lesssim$ 1500)

Name	Mass	Spin	I	I_3	Y
p^+	938	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	+1
n	940	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	+1
Λ	1116	$\frac{1}{2}$	0	0	0
Σ^+	1189	$\frac{1}{2}$	1	+1	0
Σ^0	1193	$\frac{1}{2}$	1	0	0
Σ^-	1197	$\frac{1}{2}$	1	-1	0
Δ^{++}	1231	$\frac{3}{2}$	$\frac{3}{2}$	$+\frac{3}{2}$	+1
Δ^+	1232	$\frac{3}{2}$	$\frac{3}{2}$	$+\frac{1}{2}$	+1
Δ^0	1234	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	+1
Δ^-	1235	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	+1
Ξ^0	1315	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	-1
Ξ^-	1321	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	-1
Σ^{*+}	1383	$\frac{3}{2}$	1	+1	0
Σ^{*0}	1384	$\frac{3}{2}$	1	0	0
Σ^{*-}	1387	$\frac{3}{2}$	1	-1	0
$\Lambda(1405)$	1406	$\frac{1}{2}$	0	0	0
$N^+(1440)$	1440	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	+1
$N^0(1440)$	1440	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	+1
Ξ^{*0}	1532	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	-1
Ξ^{*-}	1535	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	-1

Masses in MeV/c^2.
I = Total isospin
I_3 = Third component of isospin.
Y = hypercharge, in chapter 5

Mesons (mass < 900)

Name	Mass	Spin	I	I_3	Y
π^+	139	0	1	+1	0
π^0	135	0	1	0	0
π^-	139	0	1	+1	0
K^+	494	0	$\frac{1}{2}$	$+\frac{1}{2}$	+1
K^0	498	0	$\frac{1}{2}$	$-\frac{1}{2}$	+1
$\bar{K}^0$	498	0	$\frac{1}{2}$	$+\frac{1}{2}$	-1
K^-	494	0	$\frac{1}{2}$	$-\frac{1}{2}$	-1
η	547	0	0	0	0
ρ^+	770	1	1	+1	0
ρ^0	770	1	1	0	0
ρ^-	770	1	1	+1	0
ω	782	1	0	0	0
K^{*+}	896	1	$\frac{1}{2}$	$+\frac{1}{2}$	+1
K^{*0}	892	1	$\frac{1}{2}$	$-\frac{1}{2}$	+1
$\bar{K}^{*0}$	892	1	$\frac{1}{2}$	$+\frac{1}{2}$	-1
K^{*-}	896	1	$\frac{1}{2}$	$-\frac{1}{2}$	-1

Masses in MeV/c^2.
I = Total isospin
I_3 = Third component of isospin.
Y = hypercharge, in chapter 5

[3] According to the standard model of particle physics, there is a tenth, the Higgs boson, which is as of 2009 undiscovered. Extensions beyond the standard model predict many others.

sometimes not. For example, the π^0 is its own anti-particle, while the $\pi^\pm$ are antiparticles of each other.

If isospin is going to be a symmetry of the strong interactions, then not only must the proton and neutron fit into a representation of the group *SU*(2), so also must the rest of these particles. Indeed, it turns out that they *do* fit into such groupings, as listed in the two tables. Every one of these particles comes with partners which together form a complete set that interchange with each other according to the rules

$$\mathcal{I}_a \left| B_m^{(I)} \right\rangle = \sum_{m'} \left| B_{m'}^{(I)} \right\rangle \left(T_a^{(I)} \right)_{m'm} \tag{3.16}$$

The matrices $T_a^{(I)}$ are simply the representations found in the previous chapter of the group *SU*(2). In particular, the T_+ matrices, are given explicitly by equation (3.19). For example, for the four Δ-particles, we have

$$\mathcal{I}_+ \left|\Delta^-\right\rangle = \sqrt{3}\left|\Delta^0\right\rangle, \quad \mathcal{I}_+ \left|\Delta^0\right\rangle = 2\left|\Delta^+\right\rangle, \quad \mathcal{I}_+ \left|\Delta^+\right\rangle = \sqrt{3}\left|\Delta^{++}\right\rangle, \quad \mathcal{I}_+ \left|\Delta^{++}\right\rangle = 0 \tag{3.17}$$

We can similarly get relations for all the $\mathcal{I}_-$ operators:

$$\mathcal{I}_- \left|\Delta^{++}\right\rangle = \sqrt{3}\left|\Delta^+\right\rangle, \quad \mathcal{I}_- \left|\Delta^+\right\rangle = 2\left|\Delta^0\right\rangle, \quad \mathcal{I}_- \left|\Delta^0\right\rangle = \sqrt{3}\left|\Delta^-\right\rangle, \quad \mathcal{I}_- \left|\Delta^-\right\rangle = 0. \tag{3.18}$$

As we look at the tables of bosons and mesons, some patterns quickly emerge. Firstly, baryons are *always* fermions (half-integer spin) and mesons are *always* bosons (integer spin). This pattern continues for much more massive particles as well. Furthmore, only a small number of isospin representations are actually ever used. Baryons are isospin 0, $\frac{1}{2}$, 1, or $\frac{3}{2}$, while mesons are isospin 0, $\frac{1}{2}$, or 1. This pattern continues for heavier particles as well.

Now, the mass of a fermion is supposed to be determined by some sort of Hamiltonian, so $m_X = \left\langle X \right| H \left| X \right\rangle$. With the help of (3.17), and the Hermitian conjugate of (3.18),

$$\left\langle\Delta^{++}\right| \mathcal{I}_+ = \sqrt{3}\left\langle\Delta^+\right|, \quad \left\langle\Delta^+\right| \mathcal{I}_+ = 2\left\langle\Delta^0\right|, \quad \left\langle\Delta^0\right| \mathcal{I}_+ = \sqrt{3}\left\langle\Delta^-\right|, \quad \left\langle\Delta^-\right| \mathcal{I}_+ = 0. \tag{3.19}$$

we can show that all four Δ-particles have approximately the same mass. Assuming the Hamiltonian commutes with isospin, we have

$$\begin{aligned}
m_{\Delta^{++}} &= \left\langle\Delta^{++}\right| H \left|\Delta^{++}\right\rangle = \tfrac{1}{\sqrt{3}}\left\langle\Delta^{++}\right| H\mathcal{I}_+ \left|\Delta^+\right\rangle = \tfrac{1}{\sqrt{3}}\left\langle\Delta^{++}\right| \mathcal{I}_+ H \left|\Delta^+\right\rangle = \left\langle\Delta^+\right| H \left|\Delta^+\right\rangle = m_{\Delta^+}, \\
m_{\Delta^+} &= \left\langle\Delta^+\right| H \left|\Delta^+\right\rangle = \tfrac{1}{2}\left\langle\Delta^+\right| H\mathcal{I}_+ \left|\Delta^0\right\rangle = \tfrac{1}{2}\left\langle\Delta^+\right| \mathcal{I}_+ H \left|\Delta^0\right\rangle = \left\langle\Delta^0\right| H \left|\Delta^0\right\rangle = m_{\Delta^0}, \\
m_{\Delta^0} &= \left\langle\Delta^0\right| H \left|\Delta^0\right\rangle = \tfrac{1}{\sqrt{3}}\left\langle\Delta^0\right| H\mathcal{I}_+ \left|\Delta^-\right\rangle = \tfrac{1}{\sqrt{3}}\left\langle\Delta^0\right| \mathcal{I}_+ H \left|\Delta^-\right\rangle = \left\langle\Delta^-\right| H \left|\Delta^-\right\rangle = m_{\Delta^-}, \\
& m_{\Delta^{++}} = m_{\Delta^+} = m_{\Delta^0} = m_{\Delta^-}.
\end{aligned} \tag{3.20}$$

Similar methods will demonstrate that every set of fermionic "partners" should have the same mass. For bosons, the Hamiltonian, in a sense, gives you the mass squared instead, but the conclusion is the same, the masses must be equal.

We understand how to let isospin operators act on the baryons and mesons. Does it work the same way for the anti-baryons? According to the rules laid out in chapter one, the anti-particles should transform according to the complex conjugate representation, and as we showed there, under complex conjugation, $T_a^{(A*)} = -T_a^{(A)*}$. Since we are working with Hermitian operators, we can rewrite this as $T_a^{(A*)} = -T_a^{(A)T}$, with T denoting transpose, or the switching of rows and columns. For the raising and lowering operators, it follows that

$$T_\pm^{(A*)} = T_1^{(A*)} \pm iT_2^{(A*)} = -\left(T_1^{(A)} \pm iT_2^{(A)}\right)^* = -T_\pm^{(A)T} \quad \text{and} \quad T_3^{(A*)} = -T_3^{(A)} \tag{3.21}$$

The effect of these is that the anti-particles have the opposite T_3 values, and also $T_\pm$ has the opposite sign. Mathematically, if we have a set of baryons $\left|B_m^{(I)}\right\rangle$, then their corresponding anti-particles $\left|\bar{B}_m^{(I)}\right\rangle$ will transform according to

$$\mathcal{I}_3\left|\bar{B}_m^{(I)}\right\rangle = -m\left|\bar{B}_m^{(I)}\right\rangle \quad \text{and} \quad \mathcal{I}_\pm\left|\bar{B}_m^{(I)}\right\rangle = -\sum_{m'}\left(T_\pm^{(I)}\right)_{mm'}\left|\bar{B}_{m'}^{(I)}\right\rangle \tag{3.22}$$

For example, for the anti-proton and anti-neutron, we have

$$\begin{aligned} &\mathcal{I}_3\left|\bar{p}\right\rangle = -\tfrac{1}{2}\left|\bar{p}\right\rangle, \quad \mathcal{I}_+\left|\bar{p}\right\rangle = -\left|\bar{n}\right\rangle, \quad \mathcal{I}_-\left|\bar{p}\right\rangle = 0, \\ &\mathcal{I}_3\left|\bar{n}\right\rangle = +\tfrac{1}{2}\left|\bar{n}\right\rangle, \quad \mathcal{I}_-\left|\bar{n}\right\rangle = -\left|\bar{p}\right\rangle, \quad \mathcal{I}_+\left|\bar{n}\right\rangle = 0. \end{aligned} \tag{3.23}$$

It is, of course, possible to relable the anti-baryons in such a way that these transformations will look more like the usual transformations, since, after all, the complex conjugate of the (I) representation is still the (I) representation, but it is generally considered preferable to *not* do so, since this tends to obscure the simple relationship between particles and anti-particles. So we need to know (3.23) to deal with anti-baryons.

E. Interactions of strongly interacting particles

The neat thing about isospin is that it not only provides you with information relating the masses of particles, it also tells us about interactions. For example, consider the decays

$$\Sigma^* \to \Lambda\pi \tag{3.24}$$

where Σ^* is any of the three baryons listed in the tables above, and π is any of the three pions. This decay occurs extremely quickly, suggesting it is governed by strong interactions. It is obvious that since the Λ is neutral, the Σ^* and π must have the same charge. Is it possible to find a simple relationship between the rates for these three processes? The answer is yes. The rate for this decay will depend on a variety of factors, but since it occurs, it implies there must be some term in the strong Hamiltonian connecting these states,[4] *i.e.*, we will have a rate governed by

[4] Technically, it need not occur in the Hamiltonian, but in the time evolution operator, which is the exponential of the Hamiltonian. This must also commute with isospin.

$$\Gamma\left(\Sigma_m^* \to \Lambda\pi_m\right) \propto \left|\left\langle \Lambda\pi_m \middle| H \middle| \Sigma_m^* \right\rangle\right|^2 \tag{3.25}$$

In addition there will be kinematic factors involving the integration over all possible final state momenta. These kinematic factors will depend on the masses of the various particles, but since these are nearly identical, they should make little difference, and the proportionality in (3.25) should still apply.

To relate the matrix elements in (3.25), we can use the fact that the Hamiltonian commutes with isospin. Imagine, for example, we know that some portion of the Hamiltonian turns a $\left|\Sigma^{+*}\right\rangle$ into the appropriate particles, like this:

$$H\left|\Sigma^{*+}\right\rangle = a\left|\Lambda\pi^+\right\rangle \tag{3.26}$$

Of course, we have no idea what the constant *a* is. Now, let the isospin lowering operator act on (3.26). Since it commutes with the Hamiltonian, we have

$$H\mathcal{I}_-\left|\Sigma^{*+}\right\rangle = a\mathcal{I}_-\left|\Lambda\pi^+\right\rangle \tag{3.27}$$

Since the Λ has isospin 0, the isospin operator cannot act on it, and only acts on the pion. Looking up the appropriate matrix element for isospin 1, we therefore have

$$\sqrt{2}H\left|\Sigma^{*0}\right\rangle = \sqrt{2}a\left|\Lambda\pi^0\right\rangle \quad \text{or} \quad H\left|\Sigma^{*0}\right\rangle = a\left|\Lambda\pi^0\right\rangle \tag{3.28}$$

If we let the isospin lowering operator act again, we find

$$\sqrt{2}H\left|\Sigma^{*-}\right\rangle = \sqrt{2}a\left|\Lambda\pi^-\right\rangle \quad \text{or} \quad H\left|\Sigma^{*-}\right\rangle = a\left|\Lambda\pi^-\right\rangle \tag{3.29}$$

It follows that the relevant matrix elements are all equal,

$$\left\langle \Lambda\pi^+ \middle| H \middle| \Sigma^{*+}\right\rangle = \left\langle \Lambda\pi^0 \middle| H \middle| \Sigma^{*0}\right\rangle = \left\langle \Lambda\pi^- \middle| H \middle| \Sigma^{*-}\right\rangle = a \tag{3.30}$$

and therefore the decay rates are all equal.

$$\Gamma\left(\Sigma^{*+} \to \Lambda\pi^+\right) = \Gamma\left(\Sigma^{*0} \to \Lambda\pi^0\right) = \Gamma\left(\Sigma^{*-} \to \Lambda\pi^-\right) \tag{3.31}$$

Let's do a non-trivial example, to see how it works in more complicated situations. The two Ξ^* particles decay to a Ξ plus a pion. For example, the neutral one will decay based on an interaction of the type

$$H\left|\Xi^{*0}\right\rangle = a\left|\Xi^0\pi^0\right\rangle + b\left|\Xi^-\pi^+\right\rangle \tag{3.32}$$

because these are the only possibilities based on charge conservation. These can be related by letting $\mathcal{I}_+$ act on both sides of this expression. The $\mathcal{I}_+$ would normally act on *both* particles on the right, and on the particle on the left, but $\mathcal{I}_+\left|\Xi^{*0}\right\rangle = 0$ and $\mathcal{I}_+\left|\Xi^0\right\rangle = \mathcal{I}_+\left|\pi^+\right\rangle = 0$ because these are the highest charge states in their multiplets. We therefore have

$$0 = H\mathcal{I}_+\left|\Xi^{*0}\right\rangle = a\mathcal{I}_+\left|\Xi^0\pi^0\right\rangle + b\mathcal{I}_+\left|\Xi^-\pi^+\right\rangle = \sqrt{2}a\left|\Xi^0\pi^+\right\rangle + b\left|\Xi^0\pi^+\right\rangle \tag{3.33}$$

We see that $b = -a\sqrt{2}$, which lets us rewrite (3.32) as

$$H\left|\Xi^{*0}\right\rangle = a\left|\Xi^0\pi^0\right\rangle - a\sqrt{2}\left|\Xi^-\pi^+\right\rangle \tag{3.34}$$

If we now let $\mathcal{I}_-$ act on both sides of this equation, we have

$$\begin{gathered} H\mathcal{I}_-\left|\Xi^{*0}\right\rangle = a\mathcal{I}_-\left|\Xi^0\pi^0\right\rangle - \sqrt{2}a\mathcal{I}_-\left|\Xi^-\pi^+\right\rangle, \\ H\left|\Xi^{*-}\right\rangle = a\left|\Xi^-\pi^0\right\rangle + a\sqrt{2}\left|\Xi^0\pi^-\right\rangle - 2a\left|\Xi^-\pi^0\right\rangle = a\sqrt{2}\left|\Xi^0\pi^-\right\rangle - a\left|\Xi^-\pi^0\right\rangle. \end{gathered} \tag{3.35}$$

From equation (3.34) and (3.36), we can now obtain the ratio of the various matrix elements:

$$\left\langle\Xi^0\pi^0\middle|H\middle|\Xi^{*0}\right\rangle = -\tfrac{1}{\sqrt{2}}\left\langle\Xi^-\pi^+\middle|H\middle|\Xi^{*0}\right\rangle = \tfrac{1}{\sqrt{2}}\left\langle\Xi^0\pi^-\middle|H\middle|\Xi^{*-}\right\rangle = -\left\langle\Xi^-\pi^0\middle|H\middle|\Xi^{*-}\right\rangle \tag{3.35}$$

This implies that the relative rates for these decays is

$$\Gamma\left(\Xi^{*0}\to\Xi^0\pi^0\right) = \frac{\Gamma\left(\Xi^{*0}\to\Xi^-\pi^+\right)}{2} = \frac{\Gamma\left(\Xi^{*-}\to\Xi^0\pi^-\right)}{2} = \Gamma\left(\Xi^{*-}\to\Xi^-\pi^0\right) \tag{3.37}$$

Note that this means the middle two decays occur twice as fast as the outer ones. Note also that the total decay rate for Ξ^{*0} and Ξ^{*-} work out the same. This is a general feature of these decays, and is often useful as a quick check that we have made no error.

In the simple examples given here, there are quicker ways to get the answers. For example, in calculating rates for $\Xi^*\to\Xi\pi$, we could simply note that we must make make an isospin $I=\frac{1}{2}$ state (Ξ^*) from an isospin $I=\frac{1}{2}$ state (Ξ) and an $I = 1$ state (π). This combination of isospins is exactly what Clebsch-Gordan coefficients are for, and the rate is proportional to, for example,

$$\Gamma\left(\Xi^{*0}\to\Xi^0\pi^0\right) \propto \left|\left\langle\tfrac{1}{2},+\tfrac{1}{2}\middle|\tfrac{1}{2},1;+\tfrac{1}{2},0\right\rangle\right|^2 \tag{3.38}$$

We can similarly find the rest of the corresponding matrix elements in (3.37). Personally, because the representations involved are so small, I generally find it easier to simply work them out as I have done above, but if you find that confusing, you an try simply pulling up Clebsch-Gordan coefficients.

Decays are not the only processes where we can use isospin. For example, pions can be collided with nucleons (the neutron and proton), and we can have processes like $\pi N\to\pi N$. The same techniques will work, but are more difficult to implement in this case. The problem is that the initial state is in the isospin representation $(1)\otimes\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)\oplus\left(\frac{3}{2}\right)$, as is the final state. In consequence, the matrix elements of the form $\left\langle\pi N\middle|H\middle|\pi N\right\rangle$, does not depend on one, but *two* independent parameters *a* and *b*, corresponding to the two possible isospins of the incoming/outgoing state. This makes the computations more difficult. In such circumstances, a more sophisticated technique, called *tensor methods* can prove useful for efficiently calculating the answer. I will reserve a discussion of this technique for the next chapter in the context of the group *SU*(3).