Homework Set 4

1a. Obviously, E is the identity matrix, so $EX = XE = X$, and we don't need to check this. For the others, we have 25 multiplications still to check. The work will be skipped as much as possible, and we'll use previous results whenever we can.

$$A^2 = B^2 = C^2 = DF = FD = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E,$$

$$D^2 = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F, \quad F^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D,$$

$$AB = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D, \quad BA = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F,$$

$$AC = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F, \quad CA = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D,$$

$$BC = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D, \quad CB = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F,$$

$$\begin{aligned}
AD &= AAB = EB = B, \\
DA &= CAA = CE = C, \\
AF &= AAC = EC = C, \\
FA &= BAA = BE = B, \\
BD &= BBC = EC = C, \\
DB &= ABB = AE = A, \\
BF &= BBA = EA = A, \\
FB &= CBB = CE = C, \\
CD &= CCA = EA = A, \\
DC &= BCC = BA = B, \\
CF &= CCB = EB = B, \\
FC &= ACC = AE = A.
\end{aligned}$$

1b. We now find the matrix H, defined as

$$\begin{aligned}
H &= EE^\dagger + AA^\dagger + BB^\dagger + CC^\dagger + DD^\dagger + FF^\dagger \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 4 \end{pmatrix}
\end{aligned}$$

Since this matrix is already diagonal, we choose $U = \mathbf{1}$, the identity matrix, and then $d = H$. We then define new matrices $A_i' = d^{-\frac{1}{2}} A_i d^{\frac{1}{2}}$. This has the effect of leaving the diagonal unchanged, multiplying the top right component by $1/\sqrt{3}$, and multiplying the bottom left by $\sqrt{3}$.

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B' = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad D' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad C' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad F' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Had we chosen instead to use the matrix $U = \sigma_x$ instead, this would have taken us back to the original representation given by Tinkham. It is easy to check that this is a unitary representation.