Physics 745 - Group Theory

Solution Set 32

1. [15] The group SO(4) has six generators, which can be chosen to be

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

These can be shown to satisfy the commutation relations

$$[L_a, L_b] = i\varepsilon_{abc} L_c, \quad [L_a, K_b] = i\varepsilon_{abc} K_c, \quad [K_a, K_b] = i\varepsilon_{abc} L_c.$$

(a) [1] This group is rank two, so we can pick two of these matrices to be mutually commuting. If I pick $H_1 = L_3$, what should I pick for H_2?

Obviously, K_3 commutes with L_3, so the logical choice is $H_2 = K_3$.

(b) [5] Now, combine the remaining four operators into pairs, which I call $L_\pm$ and $K_\pm$, having the property

$$[H_1, L_\pm] = \pm L_\pm \quad \text{and} \quad [H_1, K_\pm] = \pm K_\pm$$

I'm not going to tell you how to do this, you have to guess for yourself.

The logical thing to try would be something like $L_\pm = L_1 \pm iL_2$ and $K_\pm = K_1 \pm iK_2$. In the spirit of keeping things orthonormal, I will throw in a factor of $1/\sqrt{2}$, so

$$L_\pm = \tfrac{1}{\sqrt{2}}(L_1 \pm iL_2) \quad \text{and} \quad K_\pm = \tfrac{1}{\sqrt{2}}(K_1 \pm iK_2).$$

Then it is easy to see that

$$[L_3, L_\pm] = \tfrac{1}{\sqrt{2}}[L_3, L_1 \pm iL_2] = \tfrac{1}{\sqrt{2}}(iL_2 \mp i^2 L_1) = \pm\tfrac{1}{\sqrt{2}}(L_1 \pm iL_2) = \pm L_\pm,$$

$$[L_3, K_\pm] = \tfrac{1}{\sqrt{2}}[L_3, K_1 \pm iK_2] = \tfrac{1}{\sqrt{2}}(iK_2 \mp i^2 K_1) = \pm\tfrac{1}{\sqrt{2}}(K_1 \pm iK_2) = \pm K_\pm.$$

(c) [6] Unfortunately the operators you found in part (b) probably do not have simple commutation relations with H_2. Combine $L_\pm$ with $K_\pm$ to make two new operators, which I called $E_\pm$ and $F_\pm$, such that the commutation relations will always be proportional, *i.e.*,

$$[H_1, E_\pm] \propto E_\pm, \quad [H_2, E_\pm] \propto E_\pm, \quad [H_1, F_\pm] \propto F_\pm, \quad [H_2, F_\pm] \propto F_\pm.$$

First, are we sure these don't work? Let's check them. We don't have the commutator $[K_a, L_b]$, but this is $[K_a, L_b] = -[L_b, K_a] = -i\varepsilon_{bac}K_c = i\varepsilon_{abc}K_c$. So we have

$$[K_3, L_\pm] = \tfrac{1}{\sqrt{2}}[K_3, L_1 \pm iL_2] = \tfrac{1}{\sqrt{2}}\left(iK_2 \mp i^2 K_1\right) = \pm\tfrac{1}{\sqrt{2}}\left(K_1 \pm iK_2\right) = \pm K_\pm,$$
$$[K_3, K_\pm] = \tfrac{1}{\sqrt{2}}[K_3, K_1 \pm iK_2] = \tfrac{1}{\sqrt{2}}\left(iL_2 \mp i^2 L_1\right) = \pm\tfrac{1}{\sqrt{2}}\left(L_1 \pm iL_2\right) = \pm L_\pm.$$

Nope, they didn't work. But if we add and then subtract these two formulas, we see that

$$[K_3, L_\pm + K_\pm] = \pm\left(K_\pm + L_\pm\right) \quad \text{and} \quad [K_3, L_\pm - K_\pm] = \pm\left(K_\pm - L_\pm\right).$$

If we now define

$$E_\pm = \tfrac{1}{\sqrt{2}}\left(L_\pm + K_\pm\right) = \tfrac{1}{2}\left(L_1 + K_1 \pm iL_2 \pm iK_2\right)$$
$$F_\pm = \tfrac{1}{\sqrt{2}}\left(L_\pm + K_\pm\right) = \tfrac{1}{2}\left(L_1 - K_1 \pm iL_2 \mp iK_2\right)$$

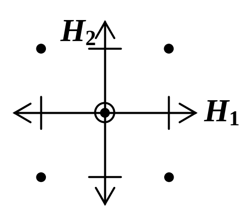
Then it is easy to see that

$$[H_1, E_\pm] = \tfrac{1}{\sqrt{2}}\left[L_3, \left(L_\pm + K_\pm\right)\right] = \pm\tfrac{1}{\sqrt{2}}\left(L_\pm + K_\pm\right) = \pm E_\pm, \quad [H_2, E_\pm] = \pm\tfrac{1}{\sqrt{2}}\left(L_\pm + K_\pm\right) = \pm E_\pm,$$
$$[H_1, E_\pm] = \tfrac{1}{\sqrt{2}}\left[L_3, \left(L_\pm - K_\pm\right)\right] = \pm\tfrac{1}{\sqrt{2}}\left(L_\pm - K_\pm\right) = \pm F_\pm, \quad [H_2, F_\pm] = \mp\tfrac{1}{\sqrt{2}}\left(L_\pm - K_\pm\right) = \mp F_\pm.$$

So we've succeeded.

(d) [3] What are the roots of this group? Make a root diagram. Don't forget the roots corresponding to H_1 and H_2!

The four non-zero roots are $(\pm 1, \pm 1)$ for the E's and $(\pm 1, \mp 1)$ for the F's. There are also two zero roots. A root diagram appears at right.



Though we weren't asked for it, the positive roots are $(1, \pm 1)$, which are also the simple roots. Since these roots are perpendicular, the Dynkin diagram is just two circles, as sketched at right. From this we can deduce that $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$.