Physics 745 - Group Theory

Solution Set 31

1. [10] The Σ^{*0} is part of the 10 of SU(3). There are four possible decays that conserve charge and hypercharge and are kinematically allowed (that means they don't violate conservation of energy and momentum): $\Sigma^0\pi^0$, $\Sigma^+\pi^-$, $\Sigma^-\pi^+$, and $\Lambda\pi^0$. Indeed, these decay modes represent the overwhelming majority of the decay modes for the Σ^{*0}.

(a) [7] In each of the four cases, work out the corresponding matrix element $\langle BM | H | \Sigma^{*0}\rangle$.

We start with equation (4.43), which says

$$\langle BM | H | B^*\rangle = a v_l^{\dagger i} u_m^{\dagger j} w^{klm} \varepsilon_{ijk}$$

We then simply plug in the appropriate numbers in each case for the matrix elements, which we obtain from equations (4.25), (4.27), and (4.28). We start by noting that the Σ^{*0} is the worst possible case; it has six different componets, all of which have indices 123 in some order and value $\frac{1}{\sqrt{6}}$. With the charged final states, there is only one term in each case, and we have

$$\begin{aligned}\langle \Sigma^+\pi^- | H | \Sigma^{*0}\rangle &= a v_1^{\dagger 2} u_2^{\dagger 1} w^{k12}\varepsilon_{21k} = a w^{312}\varepsilon_{213} = a\tfrac{1}{\sqrt{6}}(-1) = -\tfrac{1}{\sqrt{6}}a,\\ \langle \Sigma^-\pi^+ | H | \Sigma^{*0}\rangle &= a v_2^{\dagger 1} u_1^{\dagger 2} w^{k21}\varepsilon_{12k} = a w^{321}\varepsilon_{123} = a\tfrac{1}{\sqrt{6}}(1) = \tfrac{1}{\sqrt{6}}a.\end{aligned}$$

Those were the easy ones. For $\Sigma^0\pi^0$, we note that each of them is diagonal, but we can't pick the same index each time, because if we do the ε_{ijk} will vanish, so there are effectively only two terms

$$\begin{aligned}\langle \Sigma^0\pi^0 | H | \Sigma^{*0}\rangle &= a\left(v^\dagger\right)_1^1\left(u^\dagger\right)_2^2 w^{k12}\varepsilon_{12k} + a\left(v^\dagger\right)_2^2\left(u^\dagger\right)_1^1 w^{k21}\varepsilon_{21k}\\ &= a\left(\tfrac{1}{\sqrt{2}}\right)\left(-\tfrac{1}{\sqrt{2}}\right)w^{312}\varepsilon_{123} + a\left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right)w^{321}\varepsilon_{213} = -\tfrac{1}{2}a\left(\tfrac{1}{\sqrt{6}} - \tfrac{1}{\sqrt{6}}\right) = 0.\end{aligned}$$

For the $\Lambda\pi^0$, it is a little more complicated. Once again, we must choose the indices to not match, but this time that allows a lot more possibilities.

$$\begin{aligned}\langle \Lambda^0\pi^0 | H | \Sigma^{*0}\rangle &= a\left[u_1^{\dagger 1}\left(v_2^{\dagger 2} w^{k21}\varepsilon_{21k} + v_3^{\dagger 3} w^{k31}\varepsilon_{31k}\right) + u_2^{\dagger 2}\left(v_1^{\dagger 1} w^{k12}\varepsilon_{12k} + v_3^{\dagger 3} w^{k32}\varepsilon_{32k}\right)\right]\\ &= a\left[\tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{6}}w^{321}\varepsilon_{213} - \tfrac{2}{\sqrt{6}}w^{231}\varepsilon_{312}\right) - \tfrac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{6}}w^{312}\varepsilon_{123} - \tfrac{2}{\sqrt{6}}w^{132}\varepsilon_{321}\right)\right]\\ &= a\left[\tfrac{1}{\sqrt{2}}\left(-\tfrac{3}{6}\right) - \tfrac{1}{\sqrt{2}}\left(\tfrac{3}{6}\right)\right] = -\tfrac{1}{\sqrt{2}}a\end{aligned}$$

(b) [3] Which of the four "allowed" decays does not actually occur? For each of the other three cases, make a naïve prediction of the relative rate for the decay $\Gamma\left(\Sigma^{*0} \to BM\right)$, and predict the fraction that each decay occurs, which is the decay rate for a given channel divided by the total. (This is called the branching ratio. Because the Λ is noticeably lighter than the Σ's, the $\Lambda\pi^0$ mode actually is enhanced a bit compared to the naïve prediction).

Obviously, our prediction is that the $\Sigma^0\pi^0$ mode does not occur. For the other three rates, we would predict

$$\Gamma\left(\Sigma^{*0} \to \Sigma^+\pi^-\right) = \Gamma\left(\Sigma^{*0} \to \Sigma^-\pi^+\right) = \tfrac{1}{3}\Gamma\left(\Sigma^{*0} \to \Lambda^0\pi^0\right)$$

This would predict that the decay fractions are 20%, 20%, and 60%. But the $\Sigma\pi$ modes barely have enough energy to make this possible, and the actual branching fraction is about 6%, 6%, and 88%.

2. [10] The η_{c0} is a heavy, neutral, SU(3) singlet meson. Among its many decay modes, it can decay to two light mesons, $\eta_{c0} \to M'M$.

(a) [4] Suppose we write the matrix elements for the *M* and *M'* as $\left|M\right\rangle = w^i_j\left|M_i^j\right\rangle$ and $\left|M'\right\rangle = u^i_j\left|M_i^j\right\rangle$. Write down the form of all possible non-vanishing terms that appear in

$$\left\langle M'M\right|H\left|\eta_{c0}\right\rangle.$$

The η_{c0} has no indices associated with it, because it is an SU(3) singlet.

The most general form this matrix element might take would be

$$\left\langle M'M\right|H\left|\eta_{c0}\right\rangle = au_i^{\dagger j}w_j^{\dagger i}$$

This is because we have one index up and one index down, and we can't contract an index on *u* or *w* with another index on *u* or *w*. This makes it simple.

(b) [3] Calculate the relative size of the matrix element for $\left|M'M\right\rangle = \left|\pi^0\pi^0\right\rangle$, $\left|\pi^\pm\pi^\mp\right\rangle$, $\left|K^\pm K^\mp\right\rangle$, $\left|K^0\bar{K}^0\right\rangle$, $\left|\bar{K}^0 K^0\right\rangle$, and $\left|\eta\eta\right\rangle$ (eight cases in all).

This is straightforward. We simply plug everything in, and only write those terms that don't vanish.

$$\begin{aligned}
&\left\langle\pi^0\pi^0\right|H\left|\eta_{c0}\right\rangle = au_1^{\dagger 1}w_1^{\dagger 1} + au_2^{\dagger 2}w_2^{\dagger 2} = a\left(\tfrac{1}{2}+\tfrac{1}{2}\right) = a,\\
&\left\langle\pi^+\pi^-\right|H\left|\eta_{c0}\right\rangle = au_1^{\dagger 2}w_2^{\dagger 1} = a, \quad \left\langle\pi^-\pi^+\right|H\left|\eta_{c0}\right\rangle = au_2^{\dagger 1}w_1^{\dagger 2} = a,\\
&\left\langle K^+K^-\right|H\left|\eta_{c0}\right\rangle = au_1^{\dagger 3}w_3^{\dagger 1} = a, \quad \left\langle K^-K^+\right|H\left|\eta_{c0}\right\rangle = au_3^{\dagger 1}w_1^{\dagger 3} = a,\\
&\left\langle K^0\bar{K}^0\right|H\left|\eta_{c0}\right\rangle = au_2^{\dagger 3}w_3^{\dagger 2} = a, \quad \left\langle \bar{K}^0K^0\right|H\left|\eta_{c0}\right\rangle = au_3^{\dagger 2}w_2^{\dagger 3} = a,\\
&\left\langle\eta\eta\right|H\left|\eta_{c0}\right\rangle = au_1^{\dagger 1}w_1^{\dagger 1} + au_2^{\dagger 2}w_2^{\dagger 2} + au_3^{\dagger 3}w_3^{\dagger 3} = a\left(\tfrac{1}{6}+\tfrac{1}{6}+\tfrac{4}{6}\right) = a.
\end{aligned}$$

(c) [3] The mesons are so light that their relative masses are irrelevant. Predict the relative decay rates for $\Gamma\left(\eta_{c0}\to\pi^0\pi^0\right)$, $\Gamma\left(\eta_{c0}\to\pi^+\pi^-\right)$, $\Gamma\left(\eta_{c0}\to K^0\bar{K}^0\right)$, $\Gamma\left(\eta_{c0}\to K^+K^-\right)$, and $\Gamma\left(\eta_{c0}\to\eta\eta\right)$. In some cases, you will have to add the results of two different decay rates, since $\Gamma\left(A\to BC\right)$ is really the sum of $\Gamma\left(A\to BC\right)$ and $\Gamma\left(A\to CB\right)$.

The naive decay rates are that they will be all equal. Without running through the details, this isn't exactly right, because effectively some of the decays like $\Gamma\left(\eta_{c0}\to\pi^+\pi^-\right)$ are actually the sums of two decays. We therefore predict

$$\Gamma\left(\eta_{c0}\to\pi^0\pi^0\right) = \frac{\Gamma\left(\eta_{c0}\to\pi^+\pi^-\right)}{2} = \frac{\Gamma\left(\eta_{c0}\to K^0\bar{K}^0\right)}{2} = \frac{\Gamma\left(\eta_{c0}\to K^+K^-\right)}{2} = \Gamma\left(\eta_{c0}\to\eta\eta\right)$$