Physics 745 - Group Theory

Solution Set 30

1. **[5] The mass of the Ω^- can be predicted in terms of the parameters *a* and *b* from eq. (4.36).**

   (a) [2] Find the formula for the Ω^- mass in terms of *a* and *b*.

   This is straightforward:

$$m_\Omega = a w^\dagger_{ijk} w^{ijk} + b w^\dagger_{ijk} w^{ijl} \left(T_8\right)^k_l = a w^\dagger_{333} w^{333} + b w^\dagger_{333} w^{333} \left(T_8\right)^3_3 = a - \tfrac{1}{\sqrt{3}} b \,.$$

   (b) [2] Comparing with some of the other formulas, write a formula for the Ω^- mass in terms of some combination of the Δ, Σ^* and/or Ξ^* masses. There is more than one correct answer to this part.

   (c) [1] Check against the experimental value $m_\Omega = 1672\ \mathrm{MeV}/c^2$.

   If we double the Ξ^* mass, we would get

$$2m_{\Xi^*} = 2a - \tfrac{1}{\sqrt{3}} b$$

Subtracting the mass of the Ω^-, we see that

$$2m_{\Xi^*} - m_\Omega = a = m_{\Sigma^*}, \quad \text{or} \quad m_\Omega = 2m_{\Xi^*} - m_{\Sigma^*}$$

Substituting in the average for each of the other multiplets, we find

$$m_\Omega = 2m_{\Xi^*} - m_{\Sigma^*} = 2\cdot\left(1533.5\right) - 1385 = 1682\ \mathrm{MeV}/c^2$$

I missed by about 0.6%. So sue me.

2. [15] Equation (4.40) is not complete – it doesn't show where all the indices go

(a) [3] Write this equation correctly, with all the indices eliminated. You will have to have *three* coefficients in this case, which I called *a*, *b*, and *c*.

We did this in class, so it isn't that hard. We need to match up indices with down indices, and we can't match any of the factors in the tensor or the symmetry breaking T_8 to themselves, and we have

$$m_M^2 = a u^{\dagger j}{}_i u^i_j + b\left(T_8\right)^j_k u^{\dagger k}{}_i u^i_j + c\left(T_8\right)^i_k u^{\dagger j}{}_i u^k_j$$

The first term, of course, always yields a.

(b) [6] Find an equation for each of the masses m_K^2, $m_{\bar{K}}^2$, m_π^2, and m_η^2 in terms of *a*, *b*, and *c*.

We simply plug everything in. We can use any of the particles in a mutliplet to make things easy. I'll use the K, $\bar{K}$, and π^+, chosen arbitrarily. Unfortunately, the worst one is the η, where we have no choice in the matter. The a term always just gives a. We find

$$\begin{aligned}
m_K^2 &= a + b\left(T_8\right)^3_3 u^{\dagger 3}{}_2 u^2_3 + c\left(T_8\right)^2_2 u^{\dagger 3}{}_2 u^2_3 = a - \tfrac{1}{\sqrt{3}} b + \tfrac{1}{2\sqrt{3}} c, \\
m_{\bar{K}}^2 &= a + b\left(T_8\right)^2_2 u^{\dagger 2}{}_3 u^3_2 + c\left(T_8\right)^3_3 u^{\dagger 2}{}_3 u^3_2 = a + \tfrac{1}{2\sqrt{3}} b - \tfrac{1}{\sqrt{3}} c, \\
m_\pi^2 &= a + b\left(T_8\right)^2_2 u^{\dagger 2}{}_1 u^1_2 + c\left(T_8\right)^2_2 u^{\dagger 2}{}_1 u^1_2 = a + \tfrac{1}{2\sqrt{3}} b + \tfrac{1}{2\sqrt{3}} c, \\
m_\eta^2 &= a + b\left[\left(T_8\right)^1_1 u^{\dagger 1}{}_1 u^1_1 + \left(T_8\right)^2_2 u^{\dagger 2}{}_2 u^2_2 + \left(T_8\right)^3_3 u^{\dagger 3}{}_3 u^3_3\right] \\
&\quad + c\left[\left(T_8\right)^1_1 u^{\dagger 1}{}_1 u^1_1 + \left(T_8\right)^2_2 u^{\dagger 2}{}_2 u^2_2 + \left(T_8\right)^3_3 u^{\dagger 3}{}_3 u^3_3\right] \\
&= a + \left(b+c\right)\left[\tfrac{1}{2\sqrt{3}} \cdot \tfrac{1}{6} + \tfrac{1}{2\sqrt{3}} \cdot \tfrac{1}{6} - \tfrac{1}{\sqrt{3}} \cdot \tfrac{2}{3}\right] = a + \left(b+c\right)\tfrac{1-4}{6\sqrt{3}} = a - b\tfrac{1}{2\sqrt{3}} - c\tfrac{1}{2\sqrt{3}}.
\end{aligned}$$

So far, so good.

(c) [5] Find a linear equation relating these four masses, *i.e.*, eliminate *a, b,* and *c*. Arrange it so only positive coefficients appear on each side of the equation. Check it numerically.

Now things get tricky. If we add the first three, we see that only *a* appears in the sum, so we have

$$m_K^2 + m_{\bar{K}}^2 + m_\pi^2 = a - \tfrac{1}{\sqrt{3}}b + \tfrac{1}{2\sqrt{3}}c + a + \tfrac{1}{2\sqrt{3}}b - \tfrac{1}{\sqrt{3}}c + a + \tfrac{1}{2\sqrt{3}}b + \tfrac{1}{2\sqrt{3}}c = 3a.$$

Similarly, if we add the last two, we see that

$$m_\pi^2 + m_\eta^2 = a + \tfrac{1}{2\sqrt{3}}b + \tfrac{1}{2\sqrt{3}}c + a - \tfrac{1}{2\sqrt{3}}b - \tfrac{1}{2\sqrt{3}}c = 2a.$$

Combining these, we see that

$$2\left(m_K^2 + m_{\bar{K}}^2 + m_\pi^2\right) = 6a = 3\left(m_\pi^2 + m_\eta^2\right),$$
$$2m_K^2 + 2m_{\bar{K}}^2 = m_\pi^2 + 3m_\eta^2.$$

Substituting the average in for each multiplet, we want to check if

$$2(496)^2 + 2(496)^2 = 138^2 + 3(547)^2,$$
$$984064 = 916671$$

It actually failed by about 7%, I'm not sure why it's so far off.

(d) [1] An identical relationship should exist for the four masses (not masses squared) for the octuplet baryons. Check it numerically as well.

The relationship should be

$$2m_N + 2m_\Xi = m_\Sigma + 3m_\Lambda,$$
$$2\cdot 939 + 2\cdot 1318 = 1189 + 3\cdot 1116,$$
$$4514 = 4537$$

This time it worked to about half a percent error.