Homework Set 3

1. The eigenvectors are $v_\pm = \begin{pmatrix} e^{i\beta/2} \\ \pm e^{i\gamma/2} \end{pmatrix}$, with eigenvalues $\lambda_\pm = a \pm b e^{i(\beta+\gamma)/2}$. If you want them normalized, just divide by $\sqrt{2}$.

2. $N^\dagger v_\pm = \begin{pmatrix} a & be^{-i\gamma} \\ be^{-i\beta} & a \end{pmatrix}\begin{pmatrix} e^{i\beta/2} \\ \pm e^{i\gamma/2} \end{pmatrix} = \begin{pmatrix} ae^{i\beta/2} \pm be^{-i\gamma/2} \\ be^{-i\beta/2} \pm ae^{i\gamma/2} \end{pmatrix} = \left(a \pm be^{-i(\beta+\gamma)/2}\right)\begin{pmatrix} e^{i\beta/2} \\ \pm e^{i\gamma/2} \end{pmatrix} = \lambda_\pm^* v_\pm$

3. The original matrix is Hermitian if $\beta + \gamma = 0$, by inspection

4. $N^\dagger N = \begin{pmatrix} a & be^{-i\gamma} \\ be^{-i\beta} & a \end{pmatrix}\begin{pmatrix} a & be^{i\beta} \\ be^{i\gamma} & a \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ab\left(e^{i\beta} + e^{-i\gamma}\right) \\ ab\left(e^{-i\beta} + e^{i\gamma}\right) & a^2 + b^2 \end{pmatrix}$

   To be unitary, must have $a^2 + b^2 = 1$, and one of a = 0, b = 0, or $\beta + \gamma$ an odd multiple of pi.

Homework Set 4

1a. Obviously, E is the identity matrix, so $EX = XE = X$, and we don't need to check this. For the others, we have 25 multiplications still to check. The work will be skipped as much as possible, and we'll use previous results whenever we can.

$$A^2 = B^2 = C^2 = DF = FD = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E,$$

$$D^2 = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F, \quad F^2 = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D,$$

$$AB = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D, \quad BA = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F,$$

$$AC = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F, \quad CA = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D,$$

$$BC = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = D, \quad CB = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = F,$$

$$AD = AAB = EB = B,$$
$$DA = CAA = CE = C,$$
$$AF = AAC = EC = C,$$
$$FA = BAA = BE = B,$$
$$BD = BBC = EC = C,$$
$$DB = ABB = AE = A,$$
$$BF = BBA = EA = A,$$
$$FB = CBB = CE = C,$$
$$CD = CCA = EA = A,$$
$$DC = BCC = BA = B,$$
$$CF = CCB = EB = B,$$
$$FC = ACC = AE = A.$$

1b. We now find the matrix H, defined as

$$H = EE^\dagger + AA^\dagger + BB^\dagger + CC^\dagger + DD^\dagger + FF^\dagger$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 4 \end{pmatrix}$$

Since this matrix is already diagonal, we choose $U = \mathbf{1}$, the identity matrix, and then $d = H$. We then define new matrices $A_i' = d^{-\frac{1}{2}} A_i d^{\frac{1}{2}}$. This has the effect of leaving the diagonal unchanged, multiplying the top right component by $1/\sqrt{3}$, and multiplying the bottom left by $\sqrt{3}$.

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B' = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad D' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad C' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad F' = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Had we chosen instead to use the matrix $U = \sigma_x$ instead, this would have taken us back to the original representation given by Tinkham. It is easy to check that this a unitary representation.

Homework Set 5 – Tinkham 3-1

a) There are five character classes, and hence five inequivalent irreducible representations. The sums of the squares of the dimensions must equal eight, which can only be achieved if four of them are 1-dimensional, and the other one is 2-dimensional, which lets us fill in the first column of our table

b) The trivial irrep goes in the first row. For each of the three four-element normal subgroups, we get a two-element quotient group. If you think about it, you can map the "identity" subgroup to the **1** matrix, and the "other" element to the **-1** matrix. This gives us four of our representations immediately. The remaining one can be found by thinking of the matrices as rotations in 2D space, in which case you can show that G is the -1 matrix with trace -2, and all the others are traceless. You can check your results with various orthogonality conditions; for example, every column is orthogonal to every other column.

D_4	E	G	AB	CD	FH
Γ^1	1	1	1	1	1
Γ^2	1	1	1	-1	-1
Γ^3	1	1	-1	1	-1
Γ^4	1	1	-1	-1	1
Γ^5	2	-2	0	0	0

To check it, it is recommended that we take a row and multiply two columns together, and see if the result is equal to the correct resulting products. Hence, for example, if you take the column for AB and multiply it by the column for CD, and then times 4 (= 2 times 2, the number of elements in each column) you should get 4 times the FH column, times the dimensionality, where the four represents two (because there are two copies of FH) times two (because there are two elements in FH). For any class times the identity, this becomes the statement $\chi(E) n_i \chi(\mathcal{C}_i) = l n_i \chi(\mathcal{C}_i)$, and since $\chi(E) = l$, this is trivially satisfied. If you simplify the resulting expressions, you find you have to check the following for each row:

$$\begin{aligned}
&\chi(G)\chi(G) = l\chi(E) && \chi(AB)\chi(FH) = l\chi(CD) \\
&\chi(G)\chi(AB) = l\chi(AB) && \chi(CD)\chi(FH) = l\chi(AB) \\
&\chi(G)\chi(CD) = l\chi(CD) && 2\chi(AB)\chi(AB) = l\chi(E) + l\chi(G) \\
&\chi(G)\chi(FH) = l\chi(FH) && 2\chi(CD)\chi(CD) = l\chi(E) + l\chi(G) \\
&\chi(AB)\chi(CD) = l\chi(FH) && 2\chi(FH)\chi(FH) = l\chi(E) + l\chi(G)
\end{aligned}$$

These aren't too hard to check by inspection.