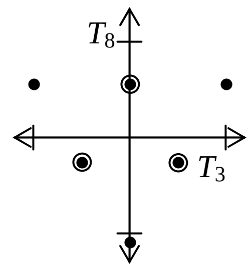
Physics 745 - Group Theory

Solution Set 29

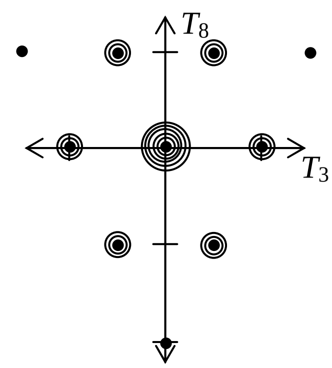
1. [5] Using a weight diagram, or tensor methods (your choice), work out the decomposition of the tensor product $3 \otimes 3$ into irreps.



If you add the three weights to themselves, you get a total of nine possible weights, as illustrated at right. The overall shape clearly corresponds to the 6 irrep, and if you remove the six corresponding weights, what is left is the $\overline{3}$ irrep. In tensor notation, we note that $v^a w^b$ has two indices up, and hence naturally forms a 6, or you can use the Levi-Civita tensor ε_{abc} to replace the two up indices with one down on, which is a $\overline{3}$. In any case, $3 \otimes 3 = 6 \oplus \overline{3}$. The dimensions work.

$$3 \otimes 3 = 6 \oplus \overline{3}$$

2. [5] Using a weight diagram, or tensor methods (your choice), work out the decomposition of the tensor product $3 \otimes 3 \otimes 3$ into irreps.



With a weight diagram, we can simply add all possible combinations of all three weights. This leads to the weight diagram above. The overall pattern is an upside down triangle of size 3, which corresponds to the 10 irrep. If you remove these weights, the next layer in is a bunch of (now) doubly degenerate weights that form a regular hexagon. This is two copies of the 8 representation. This leaves only the weight at 0, which initially had six identical weights, from which the 10 extracted one, each of the 8's extracted two, and there is therefore just one left, which corresponds to the 1 irrep.

To work this out with tensor methods is a bit trickier, though if you think of it as $3 \otimes 3 \otimes 3 = \left(6 \oplus \overline{3}\right) \otimes 3 = \left(6 \otimes 3\right) \oplus \left(\overline{3} \otimes 3\right)$, it isn't so bad. For $6 \otimes 3$, we have $u^{ij} v^k$, which could be a 10, or we can combine it with ε_{jkl} which makes something with one index down and one up, or an 8. For $\overline{3} \otimes 3$ we can have $u_i v^j$ or $u_i v^i$, which makes it an 8 or a 1. So from either argument, we have $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$. The dimensions work out.

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$