Physics 745 - Group Theory

Solution Set 22

1. [10] It is rare we will actually use the representation matrices $\Gamma^{(j)}(R)$, but occasionally it is useful. We want to work out explicitly $\Gamma^{(\frac{1}{2})}(R(\mathbf{x}))$, generated by the generators

$$T_a = \tfrac{1}{2}\sigma_{a,} \quad \text{where} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will write x in the form $\mathbf{x} = x\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector, and *x* is the magnitude of x.

(a) [3] Show that $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^2 = \mathbf{1}$, where 1 is the unit matrix, for any unit vector $\hat{\mathbf{r}}$. Furthermore, find a simplification for $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^n$ for arbitrary positive integer *n* when *n* is even or odd.

We simply write out the expression explicitly, writing $\hat{\mathbf{r}} = (r_1, r_2, r_3)$, so that

$$(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^2 = \begin{pmatrix} r_3 & r_1 - ir_2 \\ r_1 + ir_2 & -r_3 \end{pmatrix}\begin{pmatrix} r_3 & r_1 - ir_2 \\ r_1 + ir_2 & -r_3 \end{pmatrix} = \begin{pmatrix} r_3^2 + r_1^2 + r_2^2 & 0 \\ 0 & r_3^2 + r_1^2 + r_2^2 \end{pmatrix} = \mathbf{1}(\hat{\mathbf{r}}^2) = \mathbf{1}$$

Now, if you have it raised to an even power, you will simply get $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^{2n} = \mathbf{1}^n = \mathbf{1}$, while for an odd power, you will have $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^{2n+1} = \mathbf{1}^n(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}) = \hat{\mathbf{r}}\cdot\boldsymbol{\sigma}$. That's pretty simple.

(b) [4] Expand out the representation

$$\Gamma^{(\frac{1}{2})}(R(\mathbf{x})) = \exp(-i\mathbf{T}\cdot\mathbf{x}) = \sum_{n=0}^{\infty}\frac{(-i\mathbf{T}\cdot\mathbf{x})^n}{n!}$$

dividing it into even and odd terms. Simplify as much as possible

(c) [3] Show that $\Gamma^{(\frac{1}{2})}(R(\mathbf{x})) = \mathbf{1}\cos(\tfrac{1}{2}x) - i(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})\sin(\tfrac{1}{2}x)$

We simply follow the instructions, and then divide it into even and odd terms:

$$\begin{aligned}
\Gamma^{(\frac{1}{2})}(R(\mathbf{x})) &= \sum_{n=0}^{\infty}\frac{(-i\mathbf{T}\cdot\mathbf{x})^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}(-i\tfrac{1}{2}\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}x)^n \\
&= \sum_{n\text{ even}}^{\infty}\frac{1}{n!}(-i\tfrac{1}{2}x)^n(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^n + \sum_{n\text{ odd}}^{\infty}\frac{1}{n!}(-i\tfrac{1}{2}x)^n(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^n \\
&= \mathbf{1}\left[1 - \tfrac{1}{2!}(\tfrac{1}{2}x)^2 + \tfrac{1}{4!}(\tfrac{1}{2}x)^4 - + \cdots\right] + (\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})\left[-i(\tfrac{1}{2}x) + i\tfrac{1}{3!}(\tfrac{1}{2}x)^3 - i\tfrac{1}{5!}(\tfrac{1}{2}x)^5 + - \cdots\right] \\
&= \mathbf{1}\cos(\tfrac{1}{2}x) - i(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})\sin(\tfrac{1}{2}x)
\end{aligned}$$