Physics 745 - Group Theory

Solution Set 19

1. [10] In the defining representation, the four generators of the group $U(2)$ can be given by

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_4 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(a) [2] These generators are orthogonal, but not orthonormal. Assume the even ones T_2 and T_4 are normalized correctly, fix the other two by multiplying by an appropriate constant such that they will be orthonormal as well.

Our generators are supposed to have the normalization property $\mathrm{tr}(T_a T_b) = \lambda \delta_{ab}$. It isn't hard to see that you get zero if you pick $a \neq b$, but if you check the other cases, you find $\mathrm{tr}(T_1^2) = \mathrm{tr}(T_3^2) = 1$, while $\mathrm{tr}(T_2^2) = \mathrm{tr}(T_4^2) = \frac{1}{2}$. To fix this, the easiest thing to do is to let

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) [4] Using your "fixed" generators, work out all commutators $[T_a, T_b]$ for every pair $a < b$ (6 total). Write your answer in terms of other generators.

This is pretty straightforward. I sill skip the work of actually doing the commutations, we simply write out $[T_a, T_b] = T_a T_b - T_b T_a$ and simplify. We find

$$[T_1, T_2] = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\tfrac{1}{\sqrt{2}} T_4, \qquad [T_2, T_3] = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\tfrac{1}{\sqrt{2}} T_4,$$

$$[T_1, T_3] = 0, \qquad [T_2, T_4] = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\tfrac{1}{\sqrt{2}}(T_1 - T_3),$$

$$[T_1, T_4] = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\tfrac{1}{\sqrt{2}} T_2 \qquad [T_3, T_4] = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\tfrac{1}{\sqrt{2}} T_2.$$

(c) [4] Find all the non-zero structure constants f_{abc} for this group. You may use the complete anti-symmetry of f_{abc} to save work or check your answers, if you wish.

The nonzero ones can be spotted by keeping in mind that $[T_a, T_b] = i\sum_c f_{abc} T_c$. We will take advantage of the anti-symmetry on the two first indices only, and use anti-symmetry on all three as a check. By inspection, we see

$$f_{124} = -f_{214} = f_{241} = -f_{421} = -f_{142} = f_{412} = \frac{1}{\sqrt{2}} = f_{234} = -f_{324} = -f_{243} = f_{423} = f_{342} = -f_{432}\,.$$

It is clear that the results came out completely anti-symmetric.