Physics 745 - Group Theory

Solution Set 13

The original basis is:

$$\mathbf{a} = a\hat{\mathbf{x}}$$
$$\mathbf{b} = b\left(\cos\gamma\hat{\mathbf{x}} + \sin\gamma\hat{\mathbf{y}}\right)$$
$$\mathbf{c} = c\hat{\mathbf{z}}$$

An arbitrary element of the translation group is then $n_1\mathbf{a} + n_2\mathbf{b} + n_3\mathbf{c}$, where (n_1, n_2, n_3) are all integers, or instead we can choose n_1 and n_3 to be half-integers, but n_2 is still an integer.

Imagine switching basis to

$$\mathbf{T}_1 = \tfrac{1}{2}\left(\mathbf{a} + \mathbf{c}\right)$$
$$\mathbf{T}_2 = \mathbf{b}$$
$$\mathbf{T}_3 = \tfrac{1}{2}\left(\mathbf{a} - \mathbf{c}\right)$$

Then it is pretty easy to see that $\mathbf{T}_1 \pm \mathbf{T}_3$ yields the two vectors **a** and **c**. It follows that

$$n_1\mathbf{a} + n_2\mathbf{b} + n_3\mathbf{c} = n_1\left(\mathbf{T}_1 + \mathbf{T}_3\right) + n_2\mathbf{T}_2 + n_3\left(\mathbf{T}_1 - \mathbf{T}_2\right) = \left(n_1 + n_3\right)\mathbf{T}_1 + n_2\mathbf{T}_2 + \left(n_1 - n_3\right)\mathbf{T}_3$$

Given the restrictions on (n_1, n_2, n_3), it is obvious that all three of $(n_1 + n_3, n_2, n_1 - n_3)$ will be integers, and we have accomplished our goal.

The area of the parallelogram bounded by **a** and **b** is $ab\sin\gamma$, which is the same as the magnitude of $\mathbf{a}\times\mathbf{b}$. This must then be multiplied by the amount that **c** sticks out of the plane of **a** and **b**, so the total volume is $V = \left|\mathbf{a}\times\mathbf{b}\right| c\cos\theta$, where θ is the angle between the vector **c** and the perpendicular to **a** and **b**. A little thought will convince you that this implies $V = \left|\left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{c}\right|$, with the absolute value taking into account the fact that the cross product $\mathbf{a}\times\mathbf{b}$ might point the opposite direction from **c**. We therefore have

$$V = \left|\left(\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{c}\right| = \left|\left[ab\sin\gamma\hat{\mathbf{z}}\right]\cdot c\hat{\mathbf{z}}\right| = abc\sin\gamma$$
$$V' = \left|\left(\mathbf{T}_1\times\mathbf{T}_2\right)\cdot\mathbf{T}_3\right| = \tfrac{1}{4}\left|\left[\left(a\hat{\mathbf{x}} + c\hat{\mathbf{z}}\right)\times b\left(\cos\gamma\hat{\mathbf{x}} + \sin\gamma\hat{\mathbf{y}}\right)\right]\cdot\left(a\hat{\mathbf{x}} - c\hat{\mathbf{z}}\right)\right|$$
$$= \tfrac{1}{4}b\left|\left(a\sin\gamma\hat{\mathbf{z}} + c\cos\gamma\hat{\mathbf{y}} - c\sin\gamma\hat{\mathbf{x}}\right)\cdot\left(a\hat{\mathbf{x}} - c\hat{\mathbf{z}}\right)\right| = \tfrac{1}{4}abc\left|-\sin\gamma - \sin\gamma\right|$$
$$= \tfrac{1}{2}abc\sin\gamma = \tfrac{1}{2}V$$

The minus sign inside the absolute value just indicates that the basis set $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ is left-handed. If you want to, this can be fixed in numerous ways, such as changing the sign of $\mathbf{T}_3$. It is also easy to find the volume by taking the determinant of a matrix consisting of the three vectors.