Physics 745 - Group Theory

Homework Set 22

Due Monday, March 30

1. It is rare we will actually use the representation matrices $\Gamma^{(j)}(R)$, but occasionally it is useful. We want to work out explicitly $\Gamma^{(\frac{1}{2})}(R(\mathbf{x}))$, generated by the generators

$$T_a = \tfrac{1}{2}\sigma_a, \quad \text{where} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will write $\mathbf{x}$ in the form $\mathbf{x} = x\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector, and x is the magnitude of $\mathbf{x}$.

(a) Show that $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^2 = \mathbf{1}$, where $\mathbf{1}$ is the unit matrix, for any unit vector $\hat{\mathbf{r}}$.

Furthermore, find a simplification for $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})^n$ for arbitrary positive integer n when n is even or odd.

(b) Expand out the representation

$$\Gamma^{(\frac{1}{2})}(R(\mathbf{x})) = \exp(i\mathbf{T}\cdot\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(i\mathbf{T}\cdot\mathbf{x})^n}{n!}$$

dividing it into even and odd terms. Simplify as much as possible

(c) Show that

$$\Gamma^{(\frac{1}{2})}(R(\mathbf{x})) = \cos\left(\tfrac{1}{2}x\right) + i\sin\left(\tfrac{1}{2}x\right)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$$