Problems 37-40

37. Although it is likely that the "dark energy" component of the universe is a vacuum contribution with $P/\rho = -1$, other possibilities can be considered. For example, suppose that $P/\rho = w$, where *w* is a constant slightly smaller than -1. A formula for the dependence of the energy density ρ in terms of the scale factor *a* was given in class.

(a) Assume a flat universe with all matter in the form of the dark energy. Solve the Friedmann equation $\dot{a}^2/a^2 = \tfrac{8}{3}\pi G\rho$ for *a* as a function of time. You may choose your zero of time and overall scale *a* as convenient.

In class I demonstrated that one of the consequences of the Friedmann equations is that $\rho \propto a^{-3-3w}$. We therefore have $\dot{a}^2/a^2 \propto a^{-3-3w}$, or rearranging, we see that

$$
\begin{aligned}
da/dt &= Ca^{-(1+3w)/2}, \\
a^{(1+3w)/2}da &= Cdt, \\
\frac{2}{3+3w}a^{(3+3w)/2} &= Ct, \\
a(t) &= \left[\tfrac{3}{2}(1+w)Ct\right]^{2/(3+3w)}.
\end{aligned}
$$

where we chose a constant of integration equal to zero. The confusing thing about this equation is that the factor of $1+w$ is negative for $w<-1$. This implies that *t* must be negative, so you can make sense of this power. The singularity *t* = 0 is in the future.

(b) Show that if $w<-1$, the universe will expand to infinite size in finite time (the "Big Rip"). Find a formula for that time, compared to now, based on the current density of dark energy ρ_0 and the state parameter *w*.

If $w<-1$, then at *t* = 0, the exponent is negative, and therefore *a* diverges to infinity. This is the big rip.

We can figure out how far in the future this is by using the Friedmann equation,

$$
\begin{aligned}
\tfrac{8}{3}\pi G\rho = \frac{\dot{a}^2}{a^2} &= \left\{\frac{2}{3+3w}\left[\tfrac{3}{2}(1+w)Ct\right]^{2/(3+3w)-1}\tfrac{3}{2}(1+w)C\left[\tfrac{3}{2}(1+w)Ct\right]^{-2/(3+3w)}\right\}^2 \\
&= \left\{\left[\tfrac{3}{2}(1+w)Ct\right]^{-1}C\right\}^2 = \left[\tfrac{3}{2}(1+w)Ct\right]^{-2}, \\
t &= \frac{2}{3(1+w)}\left(\tfrac{8}{3}\pi G\rho\right)^{-1/2}.
\end{aligned}
$$

This is the time now. Since the big rip occurs at time 0, the time from now is just $-t$.

(c) Experimentally, the current density of dark energy is about $\frac{8}{3}\pi G\rho = \Omega_\Lambda H_0^2$, where $\Omega_\Lambda \approx 0.74$, $H_0^{-1} \approx 13.4\,\mathrm{Gyr}$, and $w = -0.97 \pm 0.07$. Assuming *w* is within one error bar of its actual value, what is the soonest the "Big Rip" could occur? (Note: This answer will be slightly off because the universe is not yet completely dominated by dark energy).

If we assume that $w > -0.97 - 0.07 = -1.04$, then the big rip comes at time

$$-t = -\frac{2}{3(1+w)}\left(\tfrac{8}{3}\pi G\rho\right)^{-1/2} = -\frac{2H_0^{-1}}{3(1+w)\sqrt{\Omega_\Lambda}} > -\frac{2(13.4\text{ Gyr})}{3(-.04)\sqrt{0.74}} \approx 260\text{ Gyr}$$

In other words, we've got a *long* time.

38. Show that the difference in the Riemann tensor between a reference metric $\hat{g}_{\mu\nu}$ and the actual metric $g_{\mu\nu}$ is given by

$$\delta R^{\ell}{}_{kji} \equiv R^{\ell}{}_{kji} - \hat{R}^{\ell}_{kji} = \hat{\nabla}_j \delta\Gamma^{\ell}_{ki} - \hat{\nabla}_i \delta\Gamma^{\ell}_{kj} + \delta\Gamma^m_{ki}\delta\Gamma^{\ell}_{mj} - \delta\Gamma^m_{kj}\delta\Gamma^{\ell}_{mi}$$

Then show the trivial consequence

$$\delta R_{ki} \equiv R_{ki} - \hat{R}^{j}_{kji} = \hat{\nabla}_j \delta\Gamma^{j}_{ki} - \hat{\nabla}_i \delta\Gamma^{j}_{kj} + \delta\Gamma^m_{ki}\delta\Gamma^{j}_{mj} - \delta\Gamma^m_{kj}\delta\Gamma^{j}_{mi}$$

We work out $R^{\ell}{}_{kji}$ by using the explicit form and substituting $\Gamma = \hat{\Gamma} + \delta\Gamma$ everywhere. We gather terms based on how many factors of $\delta\Gamma$ they have.

$$\begin{aligned}
R^{\ell}{}_{kji} &= \partial_j \Gamma^{\ell}_{ki} - \partial_i \Gamma^{\ell}_{kj} + \Gamma^m_{ki}\Gamma^{\ell}_{mj} - \Gamma^m_{kj}\Gamma^{\ell}_{mi} \\
&= \partial_j\left(\hat{\Gamma}^{\ell}_{ki} + \delta\Gamma^{\ell}_{ki}\right) - \partial_i\left(\hat{\Gamma}^{\ell}_{kj} + \delta\Gamma^{\ell}_{kj}\right) + \left(\hat{\Gamma}^m_{ki} + \delta\Gamma^m_{ki}\right)\left(\hat{\Gamma}^{\ell}_{mj} + \delta\Gamma^{\ell}_{mj}\right) - \left(\hat{\Gamma}^m_{kj} + \delta\Gamma^m_{kj}\right)\left(\hat{\Gamma}^{\ell}_{mi} + \delta\Gamma^{\ell}_{mi}\right) \\
&= \left[\partial_j \hat{\Gamma}^{\ell}_{ki} - \partial_i \hat{\Gamma}^{\ell}_{kj} + \hat{\Gamma}^m_{ki}\hat{\Gamma}^{\ell}_{mj} - \hat{\Gamma}^m_{kj}\hat{\Gamma}^{\ell}_{mi}\right] + \left[\partial_j \delta\Gamma^{\ell}_{ki} - \hat{\Gamma}^m_{kj}\delta\Gamma^{\ell}_{mi} + \hat{\Gamma}^{\ell}_{mj}\delta\Gamma^m_{ki}\right] \\
&\quad - \left[\partial_i \delta\Gamma^{\ell}_{kj} - \hat{\Gamma}^m_{ki}\delta\Gamma^{\ell}_{mj} + \hat{\Gamma}^{\ell}_{mi}\delta\Gamma^m_{kj}\right] + \left[\delta\Gamma^m_{ki}\delta\Gamma^{\ell}_{mj} - \delta\Gamma^m_{kj}\delta\Gamma^{\ell}_{mi}\right]
\end{aligned}$$

I grouped the six terms with only one factor of $\delta\Gamma$ in a suggestive way. The first expression is simply the Riemann tensor for the reference metric. The two terms in the middle are almost exactly covariant derivatives of the difference of the connections. Taking advantage of this, we see that

$$\begin{aligned}
R^{\ell}{}_{kji} &= \hat{R}^{\ell}_{kji} + \left[\hat{\nabla}_j \delta\Gamma^{\ell}_{ki} + \hat{\Gamma}^m_{ij}\delta\Gamma^{\ell}_{km}\right] - \left[\hat{\nabla}_i \delta\Gamma^{\ell}_{kj} + \hat{\Gamma}^m_{ji}\delta\Gamma^{\ell}_{km}\right] + \delta\Gamma^m_{ki}\delta\Gamma^{\ell}_{mj} - \delta\Gamma^m_{kj}\delta\Gamma^{\ell}_{mi}, \\
\delta R^{\ell}{}_{kji} &= \hat{\nabla}_j \delta\Gamma^{\ell}_{ki} - \hat{\nabla}_i \delta\Gamma^{\ell}_{kj} + \delta\Gamma^m_{ki}\delta\Gamma^{\ell}_{mj} - \delta\Gamma^m_{kj}\delta\Gamma^{\ell}_{mi}
\end{aligned}$$

To obtain the other relationship, simply set $\ell = j$ and perform the sum, which is implied by the Einstein summation convention, to obtain

$$\delta R_{ki} = \hat{\nabla}_j \delta\Gamma^{j}_{ki} - \hat{\nabla}_i \delta\Gamma^{j}_{kj} + \delta\Gamma^m_{ki}\delta\Gamma^{j}_{mj} - \delta\Gamma^m_{kj}\delta\Gamma^{j}_{mi}.$$

39. Suppose we have a metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is flat spacetime (in arbitrary coordinates), where $h_{\mu\nu}$ is small. Show that

$$R_{\mu\nu} = -\tfrac{1}{2}\hat{\nabla}_\rho \hat{\nabla}^\rho h_{\mu\nu} + \tfrac{1}{2}\hat{\nabla}_\mu \hat{\nabla}^\rho h_{\rho\nu} + \tfrac{1}{2}\hat{\nabla}_\nu \hat{\nabla}^\rho h_{\rho\mu} - \tfrac{1}{2}\hat{\nabla}_\mu \hat{\nabla}_\nu \left(\eta^{\rho\sigma} h_{\rho\sigma}\right) + O\left(h^2\right)$$

where the covariant derivatives are with respect to the flat metric, and all indices are raised and lowered with respect to the flat metric.

Obviously, we want to tackle this by treating the flat metric as the reference metric and g as the full metric. We know that $\hat{R}_{\mu\nu} = 0$, so $R_{\mu\nu} = \delta R_{\mu\nu}$. The difference between the connections, to leading order, is simply

$$\delta\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}\left(\hat{\nabla}_\nu g_{\mu\rho} + \hat{\nabla}_\mu g_{\nu\rho} - \hat{\nabla}_\rho g_{\mu\nu}\right) = \tfrac{1}{2}\eta^{\sigma\rho}\left(\hat{\nabla}_\nu h_{\mu\rho} + \hat{\nabla}_\mu h_{\nu\rho} - \hat{\nabla}_\rho h_{\mu\nu}\right) + O\left(h^2\right)$$

because the covariant derivative of η with its compatible metric vanishes. Substituting this into the expression we found in the previous problem, we have

$$\begin{aligned}
\delta R_{\mu\nu} &= \hat{\nabla}_\sigma \delta\Gamma^\sigma_{\mu\nu} - \hat{\nabla}_\mu \delta\Gamma^\sigma_{\nu\sigma} + \delta\Gamma^\sigma_{\mu\nu}\delta\Gamma^\rho_{\sigma\rho} - \delta\Gamma^\rho_{\mu\sigma}\delta\Gamma^\sigma_{\rho\nu} \\
&= \tfrac{1}{2}\left[\eta^{\sigma\rho}\hat{\nabla}_\sigma\left(\hat{\nabla}_\nu h_{\mu\rho} + \hat{\nabla}_\mu h_{\nu\rho} - \hat{\nabla}_\rho h_{\mu\nu}\right) - \eta^{\sigma\rho}\hat{\nabla}_\mu\left(\hat{\nabla}_\nu h_{\sigma\rho} + \hat{\nabla}_\sigma h_{\nu\rho} - \hat{\nabla}_\rho h_{\sigma\nu}\right)\right] + O\left(h^2\right) \\
&= \tfrac{1}{2}\left[\hat{\nabla}_\nu \hat{\nabla}^\rho h_{\rho\mu} + \hat{\nabla}_\mu \hat{\nabla}^\rho h_{\rho\nu} - \hat{\nabla}^\rho \hat{\nabla}_\rho h_{\mu\nu} - \hat{\nabla}_\mu \hat{\nabla}_\nu\left(\eta^{\sigma\rho} h_{\sigma\rho}\right)\right] + O\left(h^2\right)
\end{aligned}$$

Where we noted that the last two terms cancel. Using the fact that $R_{\mu\nu} = \delta R_{\mu\nu}$, this is the same as

$$R_{\mu\nu} = -\tfrac{1}{2}\hat{\nabla}_\rho \hat{\nabla}^\rho h_{\mu\nu} + \tfrac{1}{2}\hat{\nabla}_\mu \hat{\nabla}^\rho h_{\rho\nu} + \tfrac{1}{2}\hat{\nabla}_\nu \hat{\nabla}^\rho h_{\rho\mu} - \tfrac{1}{2}\hat{\nabla}_\mu \hat{\nabla}_\nu \left(\eta^{\rho\sigma} h_{\rho\sigma}\right) + O\left(h^2\right)$$

40. Under the same conditions as problem 39, show that

$$G_{\mu\nu} = \tfrac{1}{2}\left\{\hat{\nabla}^{\rho}\hat{\nabla}_{\mu}\overline{h}_{\rho\nu} + \hat{\nabla}^{\rho}\hat{\nabla}_{\nu}\overline{h}_{\rho\mu} - \eta_{\mu\nu}\hat{\nabla}^{\rho}\hat{\nabla}^{\sigma}\overline{h}_{\rho\sigma} - \hat{\nabla}^{\rho}\hat{\nabla}_{\rho}\overline{h}_{\mu\nu}\right\} + O\left(h^{2}\right)$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}$**.**

We first work out the Ricci scalar, which is

$$R = g^{\alpha\beta}R_{\alpha\beta} = \eta^{\alpha\beta}R_{\alpha\beta} + O\left(h^{2}\right) = -\hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right) + \hat{\nabla}^{\mu}\hat{\nabla}^{\rho}h_{\rho\mu} + O\left(h^{2}\right)$$

We therefore have

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R + O\left(h^{2}\right) \\ &= \frac{1}{2}\left\{\begin{matrix} \hat{\nabla}_{\mu}\hat{\nabla}^{\rho}h_{\rho\nu} + \hat{\nabla}_{\nu}\hat{\nabla}^{\rho}h_{\rho\mu} - \hat{\nabla}_{\rho}\hat{\nabla}^{\rho}h_{\mu\nu} - \hat{\nabla}_{\mu}\hat{\nabla}_{\nu}\left(\eta^{\rho\sigma}h_{\rho\sigma}\right) \\ +\eta_{\mu\nu}\hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right) - \eta_{\mu\nu}\hat{\nabla}^{\sigma}\hat{\nabla}^{\rho}h_{\rho\sigma} \end{matrix}\right\} + O\left(h^{2}\right) \end{aligned}$$

We now note that

$$\begin{aligned} \hat{\nabla}_{\mu}\hat{\nabla}^{\rho}\overline{h}_{\rho\nu} &= \hat{\nabla}_{\mu}\hat{\nabla}^{\rho}h_{\rho\nu} - \tfrac{1}{2}\hat{\nabla}_{\mu}\hat{\nabla}_{\nu}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right), \\ \hat{\nabla}_{\nu}\hat{\nabla}^{\rho}\overline{h}_{\rho\mu} &= \hat{\nabla}_{\nu}\hat{\nabla}^{\rho}h_{\rho\mu} - \tfrac{1}{2}\hat{\nabla}_{\nu}\hat{\nabla}_{\mu}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right), \\ \hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\overline{h}_{\mu\nu} &= \hat{\nabla}_{\rho}\hat{\nabla}^{\rho}h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right), \\ \eta_{\mu\nu}\hat{\nabla}^{\sigma}\hat{\nabla}^{\rho}\overline{h}_{\rho\sigma} &= \eta_{\mu\nu}\hat{\nabla}^{\sigma}\hat{\nabla}^{\rho}h_{\rho\sigma} - \tfrac{1}{2}\eta_{\mu\nu}\hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right), \end{aligned}$$

or, adding the first two and subtracting the last two, we see that

$$\begin{aligned} &\hat{\nabla}_{\mu}\hat{\nabla}^{\rho}\overline{h}_{\rho\nu} + \hat{\nabla}_{\nu}\hat{\nabla}^{\rho}\overline{h}_{\rho\mu} - \hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\overline{h}_{\mu\nu} - \eta_{\mu\nu}\hat{\nabla}^{\sigma}\hat{\nabla}^{\rho}\overline{h}_{\rho\sigma} = \\ &\hat{\nabla}_{\mu}\hat{\nabla}^{\rho}h_{\rho\nu} + \hat{\nabla}_{\nu}\hat{\nabla}^{\rho}h_{\rho\mu} - \hat{\nabla}_{\rho}\hat{\nabla}^{\rho}h_{\mu\nu} - \eta_{\mu\nu}\hat{\nabla}^{\sigma}\hat{\nabla}^{\rho}h_{\rho\sigma} - \hat{\nabla}_{\nu}\hat{\nabla}_{\mu}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right) - \eta_{\mu\nu}\hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\left(h_{\alpha\beta}\eta^{\alpha\beta}\right). \end{aligned}$$

Comparing with the expression for the Einstein tensor, we see that

$$G_{\mu\nu} = \tfrac{1}{2}\left\{\hat{\nabla}_{\mu}\hat{\nabla}^{\rho}\overline{h}_{\rho\nu} + \hat{\nabla}_{\nu}\hat{\nabla}^{\rho}\overline{h}_{\rho\mu} - \hat{\nabla}_{\rho}\hat{\nabla}^{\rho}\overline{h}_{\mu\nu} - \eta_{\mu\nu}\hat{\nabla}^{\sigma}\hat{\nabla}^{\rho}\overline{h}_{\rho\sigma}\right\} + O\left(h^{2}\right).$$