Solutions to Problems 32-36

32. [10] Rindler spacetime is a two-dimensional spacetime of considerable importance in the understanding of quantum field theory in curved spacetime. The metric is given by

$$ds^2 = -x^2 dt^2 + dx^2$$

(a) [3] Find all non-zero components of the connection $\Gamma^\alpha_{\mu\nu}$. Find the four-acceleration (actually, two-acceleration) A^μ and its magnitude $a^2 = A^\mu A_\mu$ experienced by an observer staying at fixed position *x*. Show that it diverges at the apparent coordinate singularity *x* = 0.

The connection contains terms proportional to the derivative of the metric. The only non-vanishing derivative is $\partial_x g_{tt} = -2x$. It follows that the only non-vanishing connection components contain two *t*'s and an *x*, and are

$$\Gamma^x_{tt} = -\tfrac{1}{2} g^{xx} \partial_x g_{tt} = -\tfrac{1}{2}(-2x) = x, \quad \Gamma^t_{tx} = \Gamma^t_{xt} = \tfrac{1}{2} g^{tt} \partial_x g_{tt} = \tfrac{1}{2}\left(-x^{-2}\right)(-2x) = x^{-1}.$$

An object at rest has $u^x = 0$, and since $u^\alpha u_\alpha = -1$, $-1 = -x^2 u^t u^t$, so $u^t = x^{-1}$. The four-acceleration is then

$$A^\mu = u^\alpha \nabla_\alpha u^\mu = u^\alpha \left(\partial_\alpha u^\mu + u^\beta \Gamma^\mu_{\beta\alpha}\right) = u^\alpha u^\beta \Gamma^\mu_{\alpha\beta} = u^t u^t \Gamma^\mu_{tt} = x^{-2} \delta^\mu_x x = x^{-1} \delta^\mu_x,$$

Its magnitude is just $a^2 = g_{\alpha\beta} u^\alpha u^\beta = g_{xx} u^x u^x = x^{-2}$. This obviously does diverge at *x* = 0.

(b) [4] Calculate a relationship between *x* and *t* for a light beam moving leftwards/rightwards. From this relationship, define two null coordinates *v* and *w*. Write the metric in terms of these coordinates. The metric should look something like $ds^2 = f(v,w)\,dv\,dw$.

A light beam travels with *ds* = 0, and therefore satisfies

$$\begin{aligned} 0 &= -x^2 dt^2 + dx^2, \\ dx/x &= \pm dt \\ \ln x &= \pm t + \text{constant} \end{aligned}$$

It therefore makes sense to define two new coordinates *v* and *w* as

$$v = \ln x + t, \quad w = \ln x - t$$

It is easy to see that

$$dv = dx/x + dt, \quad dw = dx/x - dt$$

Then we see that

$$ds^2 = (dx + x dt)(dx - x dt) = x^2 du dv = \exp(u+v)\, du dv.$$

(c) [3] Define new coordinates $v' = v'(v)$ and $w' = w'(w)$ that makes the metric look as simple as possible. Show that in these new coordinates, the former coordinate singularity at *x* = 0 has disappeared.

This is really pretty easy. It is clear that if we let $v' = \exp(v)$ and $w' = \exp(w)$, then $dv' = \exp(v)dv$ and $dw' = \exp(w)dw$, so that $ds^2 = du'dv'$. This is clearly just flat spacetime, and hence there never was a real singularity. It is also easy to see that $v'w' = \exp(v+w) = \exp(2\ln x) = x^2$. Our original coordinate singularity has been moved to $v' = w' = 0$, but there is no problem at these values.

33. [10] Consider a particle moving radially in the FRW (Friedmann-Robertson-Walker) metric, so that $u^\phi = u^\theta = 0$. I recommend using the (t, ψ, θ, ϕ) coordinates

(a) [5] Find an equation for how the radial velocity changes $du^\psi / d\tau$ for a particle following a geodesic.

The radial velocity satisfies the geodesic equation,

$$\frac{du^\psi}{d\tau} = -\Gamma^\psi_{\alpha\beta} u^\alpha u^\beta = -\tfrac{1}{2} g^{\psi\psi} \left(\partial_\alpha g_{\psi\beta} + \partial_\beta g_{\psi\alpha} - \partial_\psi g_{\alpha\beta} \right) u^\alpha u^\beta$$

The only terms that don't vanish will be ones where you take a time derivative of $g_{\psi\psi}$. We therefore have

$$\frac{du^\psi}{d\tau} = -\frac{1}{2a^2}\left(2u^\psi u^t \partial_t a^2 \right) = -\frac{2\dot{a}}{a} u^\psi u^t$$

(b) [5] Define $p = mu^{\hat{\psi}}$, where $u^{\hat{\psi}}$ is the radial velocity in orthonormal coordinates, *m* is the (constant) mass of the particle and *p* is the momentum. Show that ap is constant; that is, that $d(ap)/d\tau = 0$.

First we need to write the orthonormal velocity, which is easy. We see from the diagonal metric that $e^{\hat{\psi}}{}_\psi = \sqrt{g_{\psi\psi}} = a$, so $u^{\hat{\psi}} = e^{\hat{\psi}}{}_\psi u^\psi = au^\psi$, so that

$$\begin{aligned}
\frac{d(pa)}{d\tau} &= \frac{d}{d\tau}\left(mu^{\hat{\psi}} a \right) = m\frac{d}{dt}\left(a^2 u^\psi \right) = 2ma\frac{da}{d\tau}u^\psi + ma^2 \frac{du^\psi}{d\tau} \\
&= 2ma\frac{da}{dt}\frac{dt}{d\tau}u^\psi - 2ma^2\frac{\dot{a}}{a}u^t u^\psi = 2ma\dot{a}\left(u^t u^\psi - u^t u^\psi \right) = 0.
\end{aligned}$$

34. [10] Find the volume of the universe, defined as $V = \int \sqrt{\gamma}\, d^3\mathbf{r}$, where γ is the determinant of the space part of the metric, for a closed universe (*k* = 1). I recommend using the (t,ψ,θ,ϕ) coordinates.

(a) [5] In terms of the scale factor *a* at any time.

Working in the suggested coordinate system, our metric is

$$ds^2 = -dt^2 + a^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

We therefore see that $\sqrt{\gamma} = a^3\sin^2\psi\sin\theta$. The axial coordinate runs from 0 to 2π, while the other two coordinates run from 0 to π. We therefore have

$$V = a^3\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\pi} \sin^2\psi\, d\psi = a^3 2\pi \cdot 2 \cdot \tfrac{1}{2}\pi = 2\pi^2 a^3$$

(b) [5] At present, in terms of H_0 and Ω (assuming $\Omega > 1$).

We know from the Friedmann equations that at arbitrary time,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}G\rho - \frac{k}{a^2}$$

Evaluating this at present, the left side is Hubble's constant squared, and the first term on the right is the same thing times Ω. As a consequence, we have $H_0^2 = \Omega H_0^2 - k/a^2$. To have a closed universe, we need *k* = 1, so then we find $a^{-2} = (\Omega - 1)H_0^2$. Solving for *a* and substituting in part (a), we have

$$V = 2\pi^2 H_0^{-3}(\Omega - 1)^{-3/2}$$

35. [10] Assume a flat universe $k = 0$ that is completely dominated by either radiation or matter. A photon leaves the origin at *t* = 0 and travels radially outwards. Find the distance the photon travels by time *t* for each of the two cases.

The only relevant part of the metric is $ds^2 = -dt^2 + a^2 dr^2$. Since light follows null directions, $0 = -dt^2 + a^2 dr^2$, so $dr/dt = a^{-1}$.

Now, if we are in a radiation dominated universe, then $a \propto t^{1/2}$, say $a = bt^{1/2}$. We therefore have

$$r = \int_0^t \frac{dr}{dt'}dt' = \int_0^t \frac{dt'}{bt'^{1/2}} = 2t^{1/2}/b$$

The distance to the object is, however, not the coordinate distance *r*, but the physical distance *ar*, which is $ar = bt^{1/2}\left(2t^{1/2}/b\right) = 2t$.

For matter dominated, $a \propto t^{2/3}$ say $a = bt^{2/3}$. We therefore have

$$r = \int_0^t \frac{dr}{dt'} dt' = \int_0^t \frac{dt'}{bt'^{2/3}} = 3t^{1/3}/b$$

The distance to the object is $ar = bt^{2/3}\left(3t^{1/3}/b\right) = 3t$.

36. [10] Suppose a universe contains only radiation, but does not necessarily have $\Omega = 1$. Find a relationship between the Hubble constant H_0 and the age of the universe in terms of Ω.

We start with the standard Friedmann equation,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho - \frac{k}{a^2} = \frac{8\pi}{3} G\rho_0 \frac{a_0^4}{a^4} - \frac{k}{a^2} = H_0^2 \Omega \frac{a_0^4}{a^4} - \frac{k}{a_0^2} \frac{a_0^2}{a^2}$$

Evaluating everything now, we see that we have

$$H_0^2 = H_0^2 \Omega - \frac{k}{a_0^2},$$
$$-k/a_0^2 = H_0^2 (1 - \Omega)$$

We therefore have

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega \frac{a_0^4}{a^4} + (1-\Omega) \frac{a_0^2}{a^2} \right],$$
$$\frac{\dot{u}^2}{u^2} = H_0^2 \left(\Omega u^{-4} + (1-\Omega) u^{-2} \right),$$
$$du/dt = H_0 \left(\Omega u^{-2} + 1 - \Omega \right).$$

where we have made the standard substitution $u = a/a_0$. The age of the universe is then

$$t_0 = \int_0^1 \frac{dt}{du} du = H_0^{-1} \int_0^1 \frac{du}{\sqrt{\Omega u^{-2} + 1 - \Omega}} = H_0^{-1} \int_0^1 \frac{u du}{\sqrt{(1-\Omega) u^2 + \Omega}} = H_0^{-1} \left. \frac{\sqrt{(1-\Omega) u^2 + \Omega}}{(1-\Omega)} \right|_{u=0}^{1}$$

$$= H_0^{-1} \frac{1 - \sqrt{\Omega}}{1 - \Omega} = \frac{H_0^{-1}}{1 + \sqrt{\Omega}}.$$