

## Solutions to Problems 29-31

**29. [10] An object is dropped from rest at infinity directly into a Schwarzschild metric, which we will assume is valid all the way to  $r = 0$ .**

**(a) [4] What is the initial “energy”  $E$  and “angular momentum”  $L$ ? Calculate a formula for  $u^r = dr/d\tau$  as a function of  $r$ . Also, find a formula for  $dr/dt$ .**

Since it is initially at rest,  $u^\phi = d\phi/d\tau = 0$ , so  $L = u_\phi = r^2 \sin^2 \theta u^\phi = 0$ . Since at infinity we are effectively at rest in flat space,  $u^t = 1$  so  $u_t = -1 = -E$ , so  $E = 1$ . Since  $u_t = -1$  is conserved, we have at any distance  $dt/d\tau = u^t = g^{tt}u_t = (1 - 2GM/r)^{-1}$ . The radial velocity can be found from the formula found in class,

$$\begin{aligned} (dr/d\tau)^2 &= E^2 - 1 + (2GM/r) - (L^2/r^2)(1 - 2GM/r) = 2GM/r, \\ dr/d\tau &= -\sqrt{2GM/r}. \end{aligned}$$

The minus sign has been chosen since the object must be falling inwards. To find  $dr/dt$ , we have

$$\frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = -\left(1 - \frac{2GM}{r}\right) \sqrt{\frac{2GM}{r}}.$$

**(b) [3] Integrate the equation you found in (a) to find the proper time  $\tau$  it takes to fall from arbitrary  $r$  to the origin.**

$$\tau = \int_r^0 \frac{d\tau}{dr'} dr' = \int_0^r \sqrt{\frac{r'}{2GM}} dr' = \frac{2}{3} \sqrt{\frac{r^3}{2GM}}.$$

It is obvious that the time is finite, and there is no obvious problem with integrating through the apparent singularity at  $r' = 2GM$ .

**(c) [3] Show that the coordinate time  $t$  it takes to fall from arbitrary  $r$  to just the “event horizon” at  $2GM$  is, in contrast, infinite.**

The corresponding integral in this case is

$$t = \int_r^{2GM} \frac{dt}{dr'} dr' = \int_{2GM}^r \frac{dr'}{1 - 2GM/r'} \sqrt{\frac{r'}{2GM}}.$$

Maple can perform this integral for us, but simply focus on the region near  $r' = 2GM$ , where the integrand is approximately  $2GM/(r' - 2GM)$ , with integral  $2GM \ln(r' - 2GM)$ . This yields infinity from the lower limit. So it takes infinite time to reach  $2GM$ .

**30. [5] An incautious traveler has just passed the Schwarzschild radius, so he is now just inside  $r = 2GM$  and moving inwards,  $u^r < 0$ . Using the fact that  $u^\alpha u_\alpha = -1$ , show that even if he is allowed to accelerate, he can never stop falling in (*i.e.*, he can't have  $u^r = 0$ ), that the magnitude of  $|u^r|$  will have a minimum value as a function of  $r$ , and find the maximum proper time before his world line terminates, *i.e.*, he reaches  $r = 0$ .**

Inside  $r = 2GM$ , we have

$$-1 = u^\alpha u_\alpha = -\frac{u^r u^r}{2GM/r-1} + (2GM/r-1)u^t u^t + r^2 u^\theta u^\theta + r^2 \sin^2 \theta u^\phi u^\phi,$$

$$u^r u^r = (2GM/r-1) \left[ 1 + (2GM/r-1)u^t u^t + r^2 u^\theta u^\theta + r^2 \sin^2 \theta u^\phi u^\phi \right].$$

Every term inside the square brackets is positive. We minimize  $u^r u^r$  if we set all other components of  $u$  to zero, so we have  $|u^r| \geq \sqrt{2GM/r-1}$ . It follows that the proper time until his demise cannot be made less than

$$\tau = \int_{2GM}^0 \frac{d\tau}{dr} dr \leq \int_0^{2GM} \frac{dr}{\sqrt{2GM/r-1}} = \int_0^{\pi/2} \frac{d(2GM \sin^2 \theta)}{\sqrt{2GM/(2GM \sin^2 \theta)-1}}$$

$$= 2GM \int_0^{\pi/2} \frac{\sin \theta 2 \sin \theta \cos \theta d\theta}{\sqrt{1-\sin^2 \theta}} = 4GM \int_0^{\pi/2} \sin^2 \theta d\theta = \pi GM.$$

**31. [15] Although four-velocity doesn't exactly apply to photons, we can define an affine parameter  $\lambda$  along the photon's path, and then define  $u^\alpha = dx^\alpha/d\lambda$ . The geodesic equation for  $u^\alpha$  is the same as usual  $du^\alpha/d\lambda = -\Gamma_{\mu\nu}^\alpha u^\mu u^\nu$ , and therefore in the Schwarzschild metric,  $u_t = -E$  and  $u_\phi = L$  will still be conserved. Our goal in this problem is to find the cross-section for a photon to be absorbed by a black hole.**

**(a) [4] Use the fact that  $u^\alpha u_\alpha = 0$  for photons to find a formula for  $u^r$  for a photon as a function of  $r$ ,  $E$ ,  $L$ , and  $M$ .**

We use the same technique that we did for material particles, with the modification that  $u^\alpha u_\alpha = 0$ . As we did before, we assume the photons are moving in the  $\theta = \frac{1}{2}\pi$  plane and therefore  $u^\theta = 0$ . We therefore have

$$0 = u^\alpha u_\alpha = g^{tt} u_t u_t + g_{rr} u^r u^r + g^{\phi\phi} u_\phi u_\phi = -\frac{E^2}{1-2GM/r} + \frac{u^r u^r}{1-2GM/r} + \frac{L^2}{r^2 \sin^2(\frac{1}{2}\pi)},$$

$$(u^r)^2 = E^2 - L^2 (1-2GM/r) r^{-2} = E^2 - L^2 r^{-2} + 2GML^2 r^{-3}$$

- (b) [4] The formula you just found should have  $(u^r)^2 = \infty$  at  $r = 0$ , then it should falls for a while, and then rises again to its ultimate value at  $r = \infty$ . It therefore has a global minimum somewhere in between. Find the value of  $r$  where this occurs, and find the value of  $(u^r)^2$  there, as a function of  $L$ ,  $E$ , and  $M$ .

The equation does, indeed, go to positive infinity at small  $r$ , where the last term dominates. At large  $r$ , the first term dominates, which is a constant, but the second term, which is the next-leading term at large  $r$ , has a positive derivative, so the function is rising at large  $r$ . It follows that somewhere between 0 and infinity there must be a global minimum. It can be found by taking the derivative and setting it to zero, which gives

$$0 = 0 + 2L^2r^{-3} - 6GML^2r^{-4}, \quad r = 3GM$$

Substituting this back into our formula for  $u^r$ , we have

$$(u^r)^2 = E^2 - L^2 \left(1 - 2GM/(3GM)\right)(3GM)^{-2} = E^2 - \frac{1}{27}L^2(GM)^{-2}$$

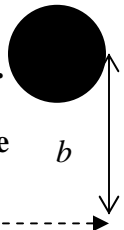
- (c) [3] If the value you found in part (b) is positive, then  $u^r$  never vanishes, which means the photon continues all the way to the singularity at  $r = 0$ . If the value you found is negative, then it must have been zero somewhere, and therefore the photon must have turned around and left again. For what values of  $L$  is the photon absorbed?

The photon is absorbed if the quantity is positive, or if  $E^2 > \frac{1}{27}L^2(GM)^{-2}$ .

Taking the square root, this is the same as  $L < \sqrt{27EGM}$ .

- (d) [4] A photon comes in from infinity, such that it has an impact parameter of  $b$ ; that is, were it not for gravity, it would miss the black hole by a distance  $b$ . What is the quantity  $L$  for this photon in terms of  $b$  and  $E$ ? (this can be calculated far away, when gravity is negligible)? Find the cross section of the black hole for photons.

photon



When we are far away, the photon will be following a straight line path, moving, of course, at the speed of light. If we set up a coordinate system as drawn, for example, then as a function of time, its position would be given by  $x = t$ ,  $y = -b$ . We can then get the azimuthal angle as a function of time, using  $\phi = \arctan(y/x) = \arctan(b/t)$ . Then we have

$$\frac{d\phi}{dt} = -\frac{-b/t^2}{1+b^2/t^2} = \frac{b}{b^2+t^2} = \frac{b}{x^2+y^2} = \frac{b}{r^2}$$

Now, we can get  $L$  using

$$L = u_\phi = g_{\phi\phi} u^\phi = r^2 \sin^2 \theta \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{dt} \frac{dt}{d\tau} = r^2 \frac{b}{r^2} E = bE$$

The photon falls into the black hole if  $L < \sqrt{27}EGM$ , or  $b < \sqrt{27}GM$ . The cross-section, therefore, is  $A = \pi b_{\max}^2 = 27\pi G^2 M^2$

**Alternative Problem: [30] Find the metric outside an electrically charged black hole.**

We need to start with some reasoning. We still want to assume spherical symmetry. The only types of fields present must be radial electric fields, so we want  $F^{\hat{t}\hat{r}} = -F^{\hat{r}\hat{t}} = E(r)$ . The metric will still be of the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We need to somehow find a formula for the electric field. This will be obtained with the help of the source-free Maxwell equation  $\nabla_\mu F^{\nu\mu} = 0$  with  $\nu = t$ . We first want to relate  $F^{\nu\mu}$  to  $F^{\hat{\nu}\hat{\mu}}$ , which requires finding an orthonormal basis. We need  $e_{\hat{\mu}}^\alpha$ , which satisfy  $e_{\hat{\mu}}^\alpha e_{\hat{\nu}}^\beta g_{\alpha\beta} = \eta_{\hat{\mu}\hat{\nu}}$ . We find that we can choose the  $e$ 's diagonal, with

$$e_{\hat{t}}^t = f^{-1/2}, \quad e_{\hat{r}}^r = h^{-1/2}, \quad e_{\hat{\theta}}^\theta = r^{-1}, \quad e_{\hat{\phi}}^\phi = r^{-1}(\sin\theta)^{-1}$$

Then we have

$$F^{tr} = -F^{rt} = F^{\hat{t}\hat{r}} (fh)^{-1/2} = \frac{E(r)}{\sqrt{f(r)h(r)}}$$

We now write out the conservation equation

$$\begin{aligned} 0 &= \nabla_\mu F^{t\mu} = \partial_\mu F^{t\mu} + F^{\alpha\mu} \Gamma_{\alpha\mu}^t + F^{t\alpha} \Gamma_{\alpha\mu}^\mu = \partial_r F^{tr} + 0 + F^{tr} \Gamma_{r\mu}^\mu \\ &= \partial_r \left( \frac{E}{\sqrt{fh}} \right) + \left( \frac{E}{\sqrt{fh}} \right) \partial_r \ln \sqrt{|g|} = \frac{\sqrt{fh} \partial_r E - E \partial_r \sqrt{fh}}{fh} + \frac{E}{\sqrt{fh}} \partial_r \ln(r^2 \sin\theta \sqrt{fh}) \\ &= \frac{\sqrt{fh} \partial_r E - r^2 E \partial_r \sqrt{fh}}{fh} + \frac{E \partial_r (r^2 \sin\theta \sqrt{fh})}{r^2 \sin\theta fh} = \frac{r^2 \partial_r E + E \partial_r r^2}{\sqrt{fh}} = \frac{\partial_r (Er^2)}{\sqrt{fh}} \end{aligned}$$

Since  $\partial_r (Er^2) = 0$ ,  $Er^2$  is a constant,  $E(r) \propto r^{-2}$ . We then define the charge in terms of the proportionality constant, so  $E(r) = Q/(4\pi\epsilon_0 r^2)$ .

Now we need to find the stress-energy tensor, which is given by

$$T^{\hat{\alpha}\hat{\beta}} = \epsilon_0 \left( F^{\hat{\alpha}}_{\hat{\mu}} F^{\hat{\beta}\hat{\mu}} - \frac{1}{4} \eta^{\hat{\alpha}\hat{\beta}} F^{\hat{\mu}\hat{\nu}} F_{\hat{\mu}\hat{\nu}} \right)$$

We first calculate

$$F^{\hat{\mu}\hat{\nu}} F_{\hat{\mu}\hat{\nu}} = F^{\hat{t}\hat{r}} F_{\hat{t}\hat{r}} + F^{\hat{r}\hat{t}} F_{\hat{r}\hat{t}} = -E^2(r) - E^2(r) = -2E^2(r)$$

We therefore have

$$\begin{aligned} T^{\hat{t}\hat{t}} &= \varepsilon_0 \left( F^{\hat{t}}_{\hat{r}} F^{\hat{r}\hat{t}} + \frac{1}{4} \eta^{\hat{t}\hat{t}} 2E^2(r) \right) = \varepsilon_0 \left( E^2(r) - \frac{1}{2} E^2(r) \right) = \frac{1}{2} \varepsilon_0 E^2(r), \\ T^{\hat{r}\hat{r}} &= \varepsilon_0 \left( F^{\hat{r}}_{\hat{t}} F^{\hat{t}\hat{r}} + \frac{1}{4} \eta^{\hat{r}\hat{r}} 2E^2(r) \right) = \varepsilon_0 \left( -E^2(r) + \frac{1}{2} E^2(r) \right) = -\frac{1}{2} \varepsilon_0 E^2(r), \\ T^{\hat{\theta}\hat{\theta}} &= T^{\hat{\phi}\hat{\phi}} = \varepsilon_0 \frac{1}{4} \eta^{\hat{\theta}\hat{\theta}} 2E^2(r) = \frac{1}{2} \varepsilon_0 E^2(r). \end{aligned}$$

Note in particular that  $T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = 0$ .

It's time to start using Einstein's equations. In particular, we find that since  $T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = 0$ , we must have

$$0 = G_{\hat{t}\hat{t}} + G_{\hat{r}\hat{r}} = \frac{h\partial_r f + f\partial_r h}{fh^2 r} = \frac{\partial_r(fh)}{fh^2 r}$$

where we have simply used the same expressions we found in class. Since  $\partial_r(fh) = 0$ ,  $fh = \text{constant}$ . If you set your clocks at infinity to be your time standard, this will work out to be  $f(r)h(r) = 1$ .

Looking now at the  $\hat{t}\hat{t}$  component of Einstein's equations, we have

$$8\pi G T_{\hat{t}\hat{t}} = G_{\hat{t}\hat{t}} = \frac{\partial_r h}{rh^2} + \frac{1}{r^2} - \frac{1}{r^2 h}$$

where again I just used the formulas I gave in class. Personally, I think it's a little easier to see what is happening if we substitute  $h = 1/f$ , then our equation becomes

$$-\frac{\partial_r f}{r} + \frac{1}{r^2} - \frac{f}{r^2} = 8\pi G T_{\hat{t}\hat{t}} = 4\pi G \varepsilon_0 E^2(r) = \frac{GQ^2}{4\pi \varepsilon_0 r^4}$$

Multiplying through by  $r^2$ , we find

$$-r\partial_r f - f + 1 = \frac{GQ^2}{4\pi \varepsilon_0 r^2}$$

The first two terms can be written succinctly as  $-\partial_r(rf)$ , so integrating, we have

$$\begin{aligned} -rf + r &= -\frac{GQ^2}{4\pi \varepsilon_0 r} + k, \\ f(r) &= 1 - \frac{k}{r} + \frac{GQ^2}{4\pi \varepsilon_0 r^2}. \end{aligned}$$

where  $k$  is an unknown constant of integration. By comparison with the weak field approximation, this constant must be  $2GM$ , where  $M$  is the mass as measured at infinity, so our metric is

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi \varepsilon_0 r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi \varepsilon_0 r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$