Solutions to Problems 25-28

25. [10] Define the first moment of the four momentum of a distribution of matter in flat spacetime as

$$V^{\alpha\beta} = \int x^\alpha T^{0\beta} d^3\vec{x}$$

(a) [5] Show that the time derivative of *V* is $\partial_0 V^{\alpha\beta} = \int T^{\alpha\beta} d^3\vec{x}$. It may be helpful to consider separately the cases $\alpha = 0$ and $\alpha = i$.

We simply take the time derivative of both sides, which yields

$$\partial_0 V^{\alpha\beta} = \int \partial_0 \left(x^\alpha T^{0\beta} \right) d^3\vec{x} = \int \left(\delta_0^\alpha T^{0\beta} + x^\alpha \partial_0 T^{0\beta} \right) d^3\vec{x} = \int \left(\delta_0^\alpha T^{0\beta} - x^\alpha \partial_i T^{i\beta} \right) d^3\vec{x}$$

where we used conservation of stress-energy, $0 = \partial_\mu T^{\mu\beta} = \partial_0 T^{0\nu} + \partial_i T^{i\nu}$, on the last term. On the final term, we now integrate by parts, ignoring the surface terms at infinity, to yield

$$\partial_0 V^{\alpha\beta} = \int \left(\delta_0^\alpha T^{0\beta} + \partial_i x^\alpha T^{i\beta} \right) d^3\vec{x} = \int \left(\delta_0^\alpha T^{0\beta} + \delta_i^\alpha T^{i\beta} \right) d^3\vec{x} = \int \delta_\mu^\alpha T^{\mu\beta} d^3\vec{x} = \int T^{\alpha\beta} d^3\vec{x}$$

(b) [5] From this, show that the angular momentum as defined in class is conserved. Also, show that the dipole moment satisfies $\vec{D}(t) = \vec{D}(0) + \vec{P}t$.

Angular momentum was defined as $S_i = \int \varepsilon_{ijk} x^j T^{0k} d^3\vec{x} = \varepsilon_{ijk} V^{jk}$. Taking the time derivative, we have $\partial_0 S_i = \varepsilon_{ijk} \partial_0 V^{jk} = \int \varepsilon_{ijk} T^{jk} d^3\vec{x}$. But the stress tensor is symmetric, while the Levi-Civita Tensor is anti-symmetric, so this vanishes, and we have $\partial_0 S_i = 0$.

The dipole moment is defined as $D^i = \int x^i T^{00} d^3\vec{x} = V^{i0}$. We therefore have $\partial_0 D^i = \int T^{i0} d^3\vec{x} = P^i$. Since momentum is conserved, the right side is constant, and integrating (and including a constant of integration) we have $\vec{D}(t) = \vec{D}(0) + \vec{P}t$.

26. [5] Show that for a stationary distribution of matter ($\partial_0 T^{\alpha\beta} = 0$),

(a) [2] $\int T^{ij} d^3\vec{x} = 0$

If $\partial_0 T^{\alpha\beta} = 0$, then any integral not involving *t* will also be constant, so in particular $\partial_0 V^{ij} = 0$. From the previous problem, this tell us that $0 = \partial_0 V^{ij} = \int T^{ij} d^3\vec{x}$.

(b) [3] $\int x^{(i} T^{j)k} d^3\vec{x} = 0$ **(hint: consider the time derivative of** $\int x^i x^j T^{0k} d^3\vec{x}$**)**

We'll take the hint. Since the stress-energy tensor is time-independent, the given integral must be time-independent, so we have

$$0 = \partial_0 \int x^i x^j T^{0k} d^3\vec{x} = \int x^i x^j \partial_0 T^{0k} d^3\vec{x} = -\int x^i x^j \partial_\ell T^{\ell k} d^3\vec{x} = \int \partial_\ell \left(x^i x^j \right) T^{\ell k} d^3\vec{x}$$
$$= \int \left(\delta^i_\ell x^j + x^i \delta^j_\ell \right) T^{\ell k} d^3\vec{x} = \int \left(x^j T^{ik} + x^i T^{jk} \right) d^3\vec{x}$$

The last expression is the desired relationship, if we divide it by two, so we have

$$\int x^{(i} T^{j)k} d^3\vec{x} = \tfrac{1}{2} \int \left(x^j T^{ik} + x^i T^{jk} \right) d^3\vec{x} = 0$$

27. [10] A two-index tensor $T^{\mu\nu}$ or $T_{\mu\nu}$ is *diagonal* if the only non-zero components have $\mu = \nu$.

(a) [5] Show that if the metric $g_{\mu\nu}$ is diagonal, then for any diagonal tensor $T^{\mu\nu}$ we will have $T^{\mu}{}_{\nu} = T^{\hat{\mu}}{}_{\hat{\nu}}$, *i.e.*, when written with one index down and one up, it will be the same thing. What is $T^{\mu}{}_{\nu}$ for a perfect fluid at rest?

If the metric is diagonal, then the distance formula looks something like

$$ds^2 = -B_0 dt^2 + \sum_{i=1}^{3} B_i dx^i dx^i$$

It is then obvious that we can pick orthonormal vectors with only diagonal components,

$$e_{\hat{\alpha}}{}^{\beta} = \delta_{\alpha}^{\beta} \big/ \sqrt{B_\alpha}\,, \quad e^{\hat{\alpha}}{}_{\beta} = \delta_{\beta}^{\alpha} \sqrt{B_\alpha}$$

For a tensor with one index up and one down, we therefore have

$$T^{\hat{\mu}}{}_{\hat{\nu}} = e^{\hat{\mu}}{}_{\alpha} e_{\hat{\nu}}{}^{\beta} T^{\alpha}{}_{\beta} = \left(\sqrt{B_\alpha}\,\delta_{\alpha}^{\mu}\right)\left(\delta_{\nu}^{\beta} \big/ \sqrt{B_\beta}\right) T^{\alpha}{}_{\beta} = T^{\mu}{}_{\nu}$$

In locally flat coordinates, a perfect fluid at rest has $T^{\hat{\mu}\hat{\nu}} = \mathrm{diag}(\rho, P, P, P)$, or lowering using $\eta_{\hat{\mu}\hat{\nu}}$, $T^{\hat{\mu}}{}_{\hat{\nu}} = \mathrm{diag}(-\rho, P, P, P)$, so we also have

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, P, P, P)$$

(b) [5] For a general spherically symmetric time-independent metric,

$$ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

use conservation of the stress-energy tensor, $\nabla_\mu T^{\mu}{}_{\nu} = 0$ with $\nu = r$ to show that

$$\partial_r P = -\tfrac{1}{2}(\rho + P)\,\partial_r(\ln f)$$

We write it out explicitly, and find

$$\begin{aligned}
0 = \nabla_\mu T^{\mu}{}_{r} &= \partial_\mu T^{\mu}{}_{r} + T^{\nu}{}_{r}\Gamma^{\mu}_{\nu\mu} - T^{\mu}{}_{\nu}\Gamma^{\nu}_{r\mu} \\
&= \partial_r T^{r}{}_{r} + T^{r}{}_{r}\left(\Gamma^{t}_{rt} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right) - T^{t}{}_{t}\Gamma^{t}_{rt} - T^{r}{}_{r}\Gamma^{r}_{rr} - T^{\theta}{}_{\theta}\Gamma^{\theta}_{r\theta} - T^{\phi}{}_{\phi}\Gamma^{\phi}_{r\phi} \\
&= \partial_r P + P\left(\Gamma^{t}_{rt} + \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right) + \rho\Gamma^{t}_{rt} - P\Gamma^{r}_{rr} - P\Gamma^{\theta}_{r\theta} - P\Gamma^{\phi}_{r\phi} \\
&= \partial_r P + (P + \rho)\Gamma^{t}_{rt} = \partial_r P + \tfrac{1}{2}(P + \rho)\, g^{tt}\left(\partial_r g_{tt} + \partial_t g_{rt} - \partial_t g_{rt}\right) \\
&= \partial_r P + (P + \rho)\frac{\partial_r f}{2f} = \partial_r P + \tfrac{1}{2}(P + \rho)\,\partial_r(\ln f)
\end{aligned}$$

The required relation then follows trivially.

28. [10] In standard coordinates, the Schwarzschild metric takes the form

$$ds^2 = -\left(1-2GM/r\right)dt^2 + \left(1-2GM/r\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Change coordinates to a new radial coordinate *R* defined by

$$r = R + GM + \frac{G^2M^2}{4R} = \left(R + \tfrac{1}{2}GM\right)^2 \Big/ R$$

Show that in this new coordinate system, the metric can be rewritten as

$$ds^2 = -A(R)dt^2 + B(R)\left[dR^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

and determine the new metric functions $A(R)$ **and** $B(R)$**.**

This is straightforward. We simply work on each term, one at a time. We start with

$$1-\frac{2GM}{r} = 1-\frac{2GMR}{\left(R+\frac{1}{2}GM\right)^2} = \frac{\left(R+\frac{1}{2}GM\right)^2 - 2GMR}{\left(R+\frac{1}{2}GM\right)^2} = \frac{R^2 - GMR + \frac{1}{4}G^2M^2}{\left(R+\frac{1}{2}GM\right)^2}$$

$$= \frac{\left(R-\frac{1}{2}GM\right)^2}{\left(R+\frac{1}{2}GM\right)^2},$$

$$dr = d\left(R + GM + \frac{G^2M^2}{4R}\right) = \left(1-\frac{G^2M^2}{4R^2}\right)dR = \frac{\left(R-\frac{1}{2}GM\right)\left(R+\frac{1}{2}GM\right)}{R^2}dR$$

We now substitute this in and hope it works out:

$$ds^2 = -\frac{\left(R-\frac{1}{2}GM\right)^2}{\left(R+\frac{1}{2}GM\right)^2}dt^2 + \frac{\left(R+\frac{1}{2}GM\right)^2}{\left(R-\frac{1}{2}GM\right)^2}\frac{\left(R-\frac{1}{2}GM\right)^2\left(R+\frac{1}{2}GM\right)^2}{R^4}dRdr^2$$

$$+\frac{\left(R+\frac{1}{2}GM\right)^2}{R^2}\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$= -\frac{\left(R-\frac{1}{2}GM\right)^2}{\left(R+\frac{1}{2}GM\right)^2}dt^2 + \frac{\left(R+\frac{1}{2}GM\right)^4}{R^4}dR^2 + \frac{\left(R+\frac{1}{2}GM\right)^4}{R^2}\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

This will clearly have the desired form providing

$$A(R) = \frac{\left(R-\frac{1}{2}GM\right)^2}{\left(R+\frac{1}{2}GM\right)^2} \quad \text{and} \quad B(R) = \frac{\left(R+\frac{1}{2}GM\right)^4}{R^4}.$$