

Solutions to Problems 21-24

21. Misner, Thorne, and Wheeler problem 16.1

The metric is $g_{\alpha\beta} = \eta_{\alpha\beta} - 2\Phi\delta_{\alpha\beta}$. Only the second term has a derivative. To leading order, the inverse metric is therefore just $g^{\alpha\beta} = \eta^{\alpha\beta}$. We need the connection coefficients, which are

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\delta} (\partial_{\alpha} g_{\beta\delta} + \partial_{\beta} g_{\alpha\delta} - \partial_{\delta} g_{\alpha\beta}) = -\eta^{\gamma\delta} (\delta_{\beta\delta} \partial_{\alpha} \Phi + \delta_{\alpha\delta} \partial_{\beta} \Phi - \delta_{\alpha\beta} \partial_{\delta} \Phi).$$

Keep in mind that the time derivatives all vanish. As a consequence, any connection coefficient with an odd number of time indices will automatically vanish, since there will inevitably be a time derivative which vanishes. The remaining non-zero components are

$$\begin{aligned} \Gamma_{i0}^0 &= \Gamma_{0i}^0 = \partial_i \Phi, & \Gamma_{00}^i &= -(-\partial_i \Phi) = \partial_i \Phi, \\ \Gamma_{jk}^i &= -\delta_{ik} \partial_j \Phi - \delta_{ij} \partial_k \Phi + \delta_{jk} \partial_i \Phi \end{aligned}$$

Since we have small velocities and nearly in flat space, we can approximate $u^{\alpha} = (1, \vec{v})$, so

$$T^{00} = \rho, \quad T^{0i} = T^{i0} = \rho v^i, \quad T^{ij} = P\delta^{ij} + \rho v^i v^j$$

In the last term, we keep $\rho v^i v^j$, because even though the velocities are small, we also anticipate that $\rho \gg P$, so we anticipate this term might not be negligible. However, whenever we have T^{ij} multiplied by Γ , it will be doubly small, and we will ignore it.

Now we write out explicitly our conservation law.

$$0 = \nabla_{\nu} T^{\mu\nu} = \partial_{\nu} T^{\mu\nu} + T^{\mu\alpha} \Gamma_{\alpha\nu}^{\nu} + T^{\alpha\nu} \Gamma_{\alpha\nu}^{\mu}$$

For $\mu = 0$, this is

$$0 = \partial_{\nu} T^{0\nu} + T^{0\alpha} \Gamma_{\alpha\nu}^{\nu} + T^{\alpha\nu} \Gamma_{\alpha\nu}^0 = \partial_0 \rho + \partial_i (\rho v^i) - 2\rho v^i \partial_i \Phi + 2\rho v^i \partial_i \Phi = d\rho/dt + \rho \partial_i v^i$$

Hence we have $d\rho/dt = -\rho \partial_i v^i$.

For $\mu = i$ we have

$$\begin{aligned} 0 &= \partial_{\nu} T^{i\nu} + T^{i\alpha} \Gamma_{\alpha\nu}^{\nu} + T^{\alpha\nu} \Gamma_{\alpha\nu}^i = \partial_0 T^{i0} + \partial_j T^{ij} + T^{00} \Gamma_{00}^i = \partial_0 (\rho v^i) + \partial_j (\rho v^j v^i + P\delta^{ij}) + \rho \partial_i \Phi \\ &= v^i (\partial_0 \rho + v^j \partial_j \rho + \rho \partial_j v^j) + (\partial_0 v^i + v^j \partial_j v^i) \rho + \partial_i P + \rho \partial_i \Phi \\ &= v^i (d\rho/dt + \rho \partial_j v^j) + \rho (dv^i/dt) + \partial_i P + \rho \partial_i \Phi \end{aligned}$$

The first term vanishes, as previously proven. The remaining terms then give

$$\rho (dv^i/dt) = -\partial_i P - \rho \partial_i \Phi$$

as expected.

22. Misner, Thorne, and Wheeler problem 16.4

To find the rate of change of the spin, we need to first define the internal angular momentum of the Earth, which in the rest frame of the Earth is given by

$$S^a = \sum_i \varepsilon^{abc} (x_{ib} - x_b) p_{ic} = \sum_i m_i \varepsilon^{abc} (x_{ib} - x_b) u_{ic}$$

where the index i represents the individual particles that compose the Earth, and the quantities without i 's represent the coordinate at the center of mass of the Earth. We have measured the position relative to the center of mass of the Earth, so as to include only the internal angular momentum, not the spin. This formula can then be generalized to an arbitrary frame by noting that in the rest frame of the Earth, $\varepsilon^{abc} = u_\delta \varepsilon^{\delta abc} = -u_\delta \varepsilon^{abc\delta}$. In this rest frame, the indices abc can then be expanded to include all coordinates with impunity, since these new terms will vanish thanks to $\varepsilon^{abc\delta}$ so we have

$$S^\alpha = -\sum_i m_i \varepsilon^{\alpha\beta\gamma\delta} (x_{i\beta} - x_\beta) u_{i\gamma} u_\delta$$

The time derivative will then be

$$dS^a/d\tau = -\sum_i m_i \varepsilon^{\alpha\beta\gamma\delta} \left[(u_{i\beta} - u_\beta) u_{i\gamma} u_\delta + (x_{i\beta} - x_\beta) (a_{i\gamma} u_\delta + u_{i\beta} a_\delta) \right]$$

The first term vanishes because it contains either $u_{i\beta} u_{i\gamma}$ or $u_\beta u_\delta$, which vanish when anti-symmetrized. On the remaining term, swap the indices $\gamma \leftrightarrow \delta$ on the last piece, to yield

$$\begin{aligned} dS^a/d\tau &= -\sum_i m_i \varepsilon^{\alpha\beta\gamma\delta} (x_{i\beta} - x_\beta) (a_{i\gamma} u_\delta - a_\gamma u_{i\delta}) \\ &= -\sum_i m_i \varepsilon^{\alpha\beta\gamma\delta} (x_{i\beta} - x_\beta) \left[(a_{i\gamma} - a_\gamma) u_\delta + a_\gamma (u_\delta - u_{i\delta}) \right] \end{aligned}$$

Now, by the way we are defining things, $(x_{i\beta} - x_\beta)$ is perpendicular to the four-velocity u , as is a_γ , and it is easy to show that for non-relativistic velocities, $(u_\delta - u_{i\delta})$ is also perpendicular to u . Hence this last term has three factors all perpendicular to u . When multiplied by $\varepsilon^{\alpha\beta\gamma\delta}$, they will have no choice but to end up parallel to (proportional to) u . Any vector k proportional to u will satisfy $-k^\alpha = k^\alpha (u_\rho u^\rho) = u^\alpha (u_\rho k^\rho)$, which allows us to rewrite the last term as

$$\begin{aligned} -\sum_i m_i \varepsilon^{\alpha\beta\gamma\delta} (x_{i\beta} - x_\beta) a_\gamma (u_\delta - u_{i\delta}) &= u^\alpha \sum_i m_i \varepsilon^{\rho\beta\gamma\delta} u_\rho (x_{i\beta} - x_\beta) a_\gamma (u_\delta - u_{i\delta}) \\ &= -u^\alpha a_\gamma \sum_i m_i \varepsilon^{\gamma\beta\delta\rho} (x_{i\beta} - x_\beta) u_{i\delta} u_\rho = u^\alpha a_\gamma S^\gamma \end{aligned}$$

For the remaining term, we assume the external gravitational fields are slowly varying over the Earth, and hence the difference in the acceleration is given by the geodesic deviation, $a_{i\gamma} - a_\gamma = R_{\gamma\sigma\nu\mu} u^\sigma u^\nu (x_i^\mu - x^\mu)$. Substituting this in, we have

$$\begin{aligned} dS^a/d\tau &= -\sum_i m_i (x_{i\beta} - x_\beta) (x_i^\mu - x^\mu) \varepsilon^{\alpha\beta\gamma\delta} R_{\gamma\sigma\nu\mu} u_\delta u^\sigma u^\nu + u^\alpha a_\gamma S^\gamma \\ &= \sum_i m_i (x_{i\beta} - x_\beta) (x_{i\mu} - x_\mu) \varepsilon^{\alpha\beta\gamma\delta} R^\mu_{\nu\sigma\gamma} u_\delta u^\nu u^\sigma + u^\alpha a_\gamma S^\gamma \end{aligned}$$

We now define the second moment of the mass distribution as

$$I_{\beta\mu} = \sum_i m_i (x_{i\beta} - x_\beta)(x_{i\mu} - x_\mu)$$

In terms of this, we now have

$$dS^\alpha/d\tau = \varepsilon^{\alpha\beta\gamma\delta} I_{\beta\mu} R^\mu{}_{\nu\gamma\sigma} u_\delta u^\nu u^\sigma + u^\alpha a_\gamma S^\gamma$$

We notice two major differences between this and the desired expression. The first is that the first term does not have the traceless part removed. We will now fix this problem. In the rest frame, we may write $I_{ij} = \mathcal{F}_{ij} + \frac{1}{3} I \delta_{ij}$, where \mathcal{F}_{ij} is the traceless part, as defined in the problem, and I is the trace. In an arbitrary frame, this will become

$$I_{\beta\mu} = \mathcal{F}_{\beta\mu} + \frac{1}{3} I (\eta_{\beta\mu} + u_\beta u_\mu)$$

Substituting this in, we have

$$dS^\alpha/d\tau = \varepsilon^{\alpha\beta\gamma\delta} \left[\mathcal{F}_{\beta\mu} + \frac{1}{3} I (\eta_{\beta\mu} + u_\beta u_\mu) \right] R^\mu{}_{\nu\gamma\sigma} u_\delta u^\nu u^\sigma + u^\alpha a_\gamma S^\gamma$$

The $u_\beta u_\mu$ term will yield a contribution $R_{\mu\nu\gamma\sigma} u^\mu u^\nu = 0$, and the $\eta_{\beta\mu}$ term yields an expression that contains $R_{\beta\nu\gamma\sigma} u^\nu u^\sigma$, which from the symmetries of the Riemann tensor can be shown to be symmetric, and hence will vanish when multiplied by $\varepsilon^{\alpha\beta\gamma\delta}$. Hence our expression simplifies to

$$dS^\alpha/d\tau = \varepsilon^{\alpha\beta\gamma\delta} \mathcal{F}_{\beta\mu} R^\mu{}_{\nu\gamma\sigma} u_\delta u^\nu u^\sigma + u^\alpha a_\gamma S^\gamma$$

The first term has now been simplified to the desired relationship. The remaining term is not incorrect, it must be there, since only with it there can we show that $u_\alpha S^\alpha$ is constant. Since I don't really understand what the funky derivative $DS^\alpha/D\tau$ means, I can't quite make sense of their equation, but I'm pretty sure my equation is right, and furthermore, it contains an effect that is not included in theirs.

Can we ignore the offending term relative to the other term? We will later address the first term. Assume the Earth moves at constant speed around the Sun, so that a_γ is purely spatial and has a magnitude of order the speed of the Earth around the Sun times the angular velocity. Assuming we focus on the spatial parts of S , as seems sensible, this tells us that $dS^\alpha/d\tau \sim v^2 \omega S^\gamma$. The velocity of the Earth around the Sun is around 10^{-4} , and the angular velocity is about $2\pi \text{ y}^{-1}$, so this implies that the Earth would precess on time scales of order $10^8 / \pi \text{ y}$. Since this exceeds the actual precession time, which is about three orders of magnitude smaller, we can pretty much ignore it. But I think it's there!

Now, we need to substitute in the expression for the specific case of the Earth. Since the velocities are small, we approximate this as

$$dS^a/d\tau = -\varepsilon^{a\beta\gamma 0} \mathcal{F}_{\beta\mu} R^\mu{}_{0\gamma 0} = -\varepsilon^{abc} \mathcal{F}_{bd} R^d{}_{0c0}$$

Using the suggested substitution, this is

$$dS^a/dt = -\varepsilon^{abc} F_{bd} \partial_c \partial_d \Phi$$

We now wish to compare this to the Newtonian theory. In the Newtonian theory, the change is due to torques. We have

$$dS^a/dt = \sum_i \varepsilon^{abc} (x_{ib} - x_b) F_{ic} = -\sum_i \varepsilon^{abc} (x_{ib} - x_b) m_i \partial_c \Phi_i$$

Now, on the final derivative, this derivative will be close to the value at the center of the Earth, so we Taylor expand the derivative around this point and write

$$dS^a/dt = -\sum_i \varepsilon^{abc} (x_{ib} - x_b) m_i (\partial_c \Phi + (x_{id} - x_d) \partial_d \partial_c \Phi)$$

By definition, the center of mass is the point such that $\sum_i (x_{ib} - x_b) m_i = 0$, so this becomes

$$dS^a/dt = -\sum_i \varepsilon^{abc} m_i (x_{ib} - x_b) (x_{id} - x_d) \partial_d \partial_c \Phi = -I_{bd} \varepsilon^{abc} \partial_d \partial_c \Phi$$

It is then a simple matter, in a manner similar to before, to show that the trace does not contribute, so we have $dS^a/dt = -\varepsilon^{abc} F_{bd} \partial_d \partial_c \Phi$, the same as before.

I refuse to do part (d) of the problem.

23. Misner, Thorne, and Wheeler problem 18.6

I don't know what the star is on the affine parameter, so we'll ignore it. We have the connection coefficients from problem 21, so let's use them:

$$\begin{aligned}\frac{dp^0}{d\lambda} &= -\Gamma_{\beta\gamma}^0 p^\beta p^\gamma = -2\Gamma_{0i}^0 p^0 p^i = -2p^0 p^i \partial_i \Phi \\ \frac{dp^i}{d\lambda} &= -\Gamma_{\beta\gamma}^i p^\beta p^\gamma = -\Gamma_{00}^i p^0 p^0 - \Gamma_{jk}^i p^j p^k = -(p^0)^2 \partial_i \Phi + (\delta_{ik} \partial_j \Phi + \delta_{ij} \partial_k \Phi - \delta_{jk} \partial_i \Phi) p^j p^k \\ &= 2p^i p^j \partial_j \Phi - (p^j p^j + p^0 p^0) \partial_i \Phi\end{aligned}$$

In a nearly flat universe, the energy $0 = g_{\mu\nu} p^\mu p^\nu \approx -p^0 p^0 + p^j p^j$, so the two terms at the end can be combined.

Now, for a photon that initially is at $y = b$ and is moving along the x -direction, the only initial components of the momentum will be $p^0 = p^x$. Since we are working in weak gravity, the momentum will only change slightly, and the path will be a straight line, so to leading order we will have

$$\begin{aligned}\frac{dp^y}{d\lambda} &= 2p^y p^j \partial_j \Phi - (p^j p^j + p^0 p^0) \partial_y \Phi = -2p^x p^x \partial_y \Phi = -2p^x \frac{dx}{d\lambda} \frac{\partial}{\partial y} \left(-\frac{GM}{r} \right) \\ &= -2p^x \frac{dx}{d\lambda} \left(\frac{GM}{r^2} \right) \frac{\partial r}{\partial y} = -2p^x \frac{dx}{d\lambda} \left(\frac{GM}{r^2} \right) \frac{y}{\sqrt{x^2 + y^2 + z^2}}\end{aligned}$$

Of course, $z = 0$ everywhere, so that's trivial, and we have $r = \sqrt{x^2 + y^2} = \sqrt{x^2 + b^2}$, so

$$\frac{dp^y}{d\lambda} = -2p^x \frac{GMb}{(x^2 + b^2)^{3/2}} \frac{dx}{d\lambda}$$

We integrate both sides over the affine parameter, treating p^x as constant, so we have

$$\begin{aligned}\Delta p^y &= -2p^x GMb \int_{-\infty}^{\infty} \frac{dx}{(x^2 + b^2)^{3/2}} = -2p^x GMb \int_{-\pi/2}^{\pi/2} \frac{b \sec^2 \theta d\theta}{(b^2 \tan^2 \theta + b^2)^{3/2}} \\ &= -2p^x \frac{GM}{b} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = -\frac{4GM}{b} p^x\end{aligned}$$

The angular deflection is $\Delta p^y / p^x = -4GM/b$. For the Sun, as listed in the appendix, $GM_\odot = 1.48 \text{ km}$ and $R_\odot = 696,000 \text{ km}$, so the angular deflection is

$$\frac{\Delta p^y}{p^x} = -\frac{4(1.48 \text{ km}) R_\odot}{696,000 \text{ km } b} \frac{180^\circ}{\pi} \frac{3600''}{1^\circ} = 1.75'' \left(\frac{R_\odot}{b} \right)$$

24. Misner, Thorne, and Wheeler problem 18.7

Although the formula given is correct, it seems unnecessarily complicated. I found it easier to just work out the fact that

$$\begin{aligned}\frac{dp_\alpha}{d\lambda} &= \frac{d}{d\lambda}(g_{\alpha\beta}p^\beta) = \frac{dg_{\alpha\beta}}{d\lambda}p^\beta + g_{\alpha\beta}\frac{dp^\beta}{d\lambda} = \frac{dx^\rho}{d\lambda}\partial_\rho g_{\alpha\beta}p^\beta - g_{\alpha\beta}\Gamma_{\rho\tau}^\beta p^\rho p^\tau \\ &= p^\rho p^\beta \partial_\rho g_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g^{\beta\sigma}(\partial_\rho g_{\tau\sigma} + \partial_\tau g_{\rho\sigma} - \partial_\sigma g_{\rho\tau})p^\rho p^\tau \\ &= p^\rho p^\beta \partial_\rho g_{\alpha\beta} - p^\rho p^\tau \partial_\rho g_{\tau\alpha} + \frac{1}{2}p^\rho p^\tau \partial_\alpha g_{\rho\tau} = \frac{1}{2}p^\rho p^\tau \partial_\alpha g_{\rho\tau}.\end{aligned}$$

It is now a moment's work to notice that since our metric is time independent (at least on the time scales we are interested in), we have $dp_0/d\lambda = 0$. Note that this argument is exact, and requires no appeal to linearized gravity.

Now, to complete the problem, imagine we are sitting next to an atom. We see an energy E that the atom is emitting. This means that in commoving orthonormal coordinates, we have $p^{\hat{0}} = E_e = p^{\hat{0}}u^{\hat{0}} = -p_{\hat{a}}u^{\hat{a}} = -p_\alpha u^\alpha$. Switching to ordinary Cartesian coordinates, $u^0 = 1 + GM/r$, so $p_0 = -E_e/(1 + GM/r)$. This will also be the value at which it is received, *i.e.*, $-E_r = -E_e/(1 + GM/r)$. Since wavelength is given by $\lambda = 2\pi/E$, we find

$$\lambda_r = 2\pi/E_r = 2\pi(1 + GM/r)/E_e = \lambda_e(1 + GM/r)$$

Or in summary

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{GM}{r}$$

Substituting the numbers for the surface of the Sun,

$$z = \frac{GM_\odot}{R_\odot} = \frac{1.48 \text{ km}}{696,000 \text{ km}} = 2.1 \times 10^{-6}.$$