Solutions to Problems 16-20

16. [5] Show that $\left[\nabla_a, \nabla_b\right] w_c = -w_d R^d{}_{cab}$**.**

We proceed exactly as we did in class for the vector case. We first do the covariant derivatives in one order, then subtract.

$$\begin{aligned}
\nabla_a \nabla_b w_c &= \partial_a \left(\nabla_b w_c\right) - \left(\nabla_b w_e\right)\Gamma^e_{ca} - \left(\nabla_e w_c\right)\Gamma^e_{ba} \\
&= \partial_a \left(\partial_b w_c - w_d \Gamma^d_{cb}\right) - \left(\partial_b w_e - w_d \Gamma^d_{eb}\right)\Gamma^e_{ca} - \left(\nabla_e w_c\right)\Gamma^e_{ba} \\
&= \partial_a \partial_b w_c - \left(\partial_a w_e\right)\Gamma^e_{cb} - \left(\partial_b w_e\right)\Gamma^e_{ca} - w_d \partial_a \Gamma^d_{cb} + w_d \Gamma^d_{eb}\Gamma^e_{ca} - \left(\nabla_e w_c\right)\Gamma^e_{ba}, \\
\nabla_b \nabla_a w_c &= \partial_b \partial_a w_c - \left(\partial_b w_e\right)\Gamma^e_{ca} - \left(\partial_a w_e\right)\Gamma^e_{cb} - w_d \partial_b \Gamma^d_{ca} + w_d \Gamma^d_{ea}\Gamma^e_{cb} - \left(\nabla_e w_c\right)\Gamma^e_{ab},
\end{aligned}$$

where in the last step we simply swapped the two indices *a* and *b*. Subtracting these two expressions, we see that the first terms and last terms cancel, while the second term cancels the third and vice versa. So we have

$$\left[\nabla_a, \nabla_b\right] w_c = -w_d \left(\partial_a \Gamma^d_{cb} - \partial_b \Gamma^d_{ca} + \Gamma^d_{ea}\Gamma^e_{cb} - \Gamma^d_{eb}\Gamma^e_{ca}\right) = -w_d R^d{}_{cab}$$

17. [5] Using the fact that $\left[\nabla_a, \nabla_b\right] g_{cd} = 0$**, prove that** $R_{cdab} = -R_{dcab}$

This is straightforward:

$$\begin{aligned}
0 = \left[\nabla_a, \nabla_b\right] g_{cd} &= g_{ed} R^e{}_{cab} + g_{ce} R^e{}_{dab} = R_{dcab} + R_{cdab}, \\
R_{cdab} &= -R_{dcab}.
\end{aligned}$$

18. [10] Show that

(a) [5] $\left[\left[\nabla_a, \nabla_b\right], \nabla_c\right] + \left[\left[\nabla_b, \nabla_c\right], \nabla_a\right] + \left[\left[\nabla_c, \nabla_a\right], \nabla_b\right] = 0$**, no matter what it is acting on. Just write it out, and it will be trivial.**

No cleverness is required, we simply write it out.

$$\begin{aligned}
&\left[\left[\nabla_a, \nabla_b\right], \nabla_c\right] + \left[\left[\nabla_b, \nabla_c\right], \nabla_a\right] + \left[\left[\nabla_c, \nabla_a\right], \nabla_b\right] \\
&\quad = \left[\nabla_a, \nabla_b\right]\nabla_c - \nabla_c\left[\nabla_a, \nabla_b\right] + \left[\nabla_b, \nabla_c\right]\nabla_a - \nabla_a\left[\nabla_b, \nabla_c\right] + \left[\nabla_c, \nabla_a\right]\nabla_b - \nabla_b\left[\nabla_c, \nabla_a\right] \\
&\quad = \nabla_a\nabla_b\nabla_c - \nabla_b\nabla_a\nabla_c - \nabla_c\nabla_a\nabla_b + \nabla_c\nabla_b\nabla_a + \nabla_b\nabla_c\nabla_a - \nabla_c\nabla_b\nabla_a - \nabla_a\nabla_b\nabla_c + \nabla_a\nabla_c\nabla_b \\
&\qquad\qquad + \nabla_c\nabla_a\nabla_b - \nabla_a\nabla_c\nabla_b - \nabla_b\nabla_c\nabla_a + \nabla_b\nabla_a\nabla_c \\
&\quad = 0
\end{aligned}$$

(b) [5] Let the operator in part (a) act on an arbitrary scalar function *f*, and show that this implies $R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca} = 0$.

This time we find

$$\begin{aligned}0 &= \left[\left[\nabla_a, \nabla_b\right], \nabla_c\right] f + \left[\left[\nabla_b, \nabla_c\right], \nabla_a\right] f + \left[\left[\nabla_c, \nabla_a\right], \nabla_b\right] f \\ &= \left[\nabla_a, \nabla_b\right]\nabla_c f - \nabla_c\left[\nabla_a, \nabla_b\right] f + \left[\nabla_b, \nabla_c\right]\nabla_a f - \nabla_a\left[\nabla_b, \nabla_c\right] f + \left[\nabla_c, \nabla_a\right]\nabla_b f \\ &\quad - \nabla_b\left[\nabla_c, \nabla_a\right] f \\ &= -\left(\nabla_d f\right) R^d{}_{cab} - 0 - \left(\nabla_d f\right) R^d{}_{abc} - 0 - \left(\nabla_d f\right) R^d{}_{bca} - 0 = -\left(\partial_d f\right)\left(R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca}\right)\end{aligned}$$

Since this must be true for any function *f*, we have

$$0 = R^d{}_{abc} + R^d{}_{bca} + R^d{}_{cab}$$

19. [15] In this problem we will consider the Riemann tensor in two dimensions

(a) [3] In 2D it is know that the Riemann tensor can be written $R_{abcd} = \frac{1}{2}\bar{R}\varepsilon_{ab}\varepsilon_{cd}$. Argue why this is reasonable.

At any given point, the Riemann tensor must be anti-symmetric on its last two indices. Since there are only two indices, these indices must be either *ab* = 12 or 21, and interchanging them must yield a minus sign. This suggests that this is proportional to ε_{cd}. In a similar way, we can argue from the anti-symmetry of the first two terms that it must be proportional to ε_{ab}. Hence $R_{abcd} \propto \varepsilon_{ab}\varepsilon_{cd}$. It isn't obvious why we should call the proportionality constant $\frac{1}{2}\bar{R}$ (which is actually a function), but certainly there is nothing wrong with it.

Note that since both sides of $R_{abcd} \propto \varepsilon_{ab}\varepsilon_{cd}$ are tensors, it is pretty clear that $\frac{1}{2}\bar{R}$ must be a tensor with no indices, *i.e.*, a scalar function. This formula can also be rewritten in the form $R_{abcd} = \frac{1}{2}\bar{R}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$.

(b) [4] Compute the Ricci tensor in terms of $\bar{R}$.

Obviously, the Ricci tensor is just $R_{ab} = g^{cd}R_{cadb} = \frac{1}{2}\bar{R}g^{cd}\varepsilon_{ca}\varepsilon_{db} = \frac{1}{2}\bar{R}\varepsilon^d{}_a\varepsilon_{db}$. We can simplify this by using the rule that two ε's with all but one index contracted is nothing more than a *g* with indices in the appropriate places, and with a factor of $(N-1)!$, where *N* is the number of dimensions. So $R_{ab} = \frac{1}{2}\bar{R}g_{ab}$

(c) [3] Show that $\bar{R}$ is the Ricci scalar.

The Ricci scalar is $R = R^{ab}R_{ab} = \frac{1}{2}\bar{R}g^{ab}g_{ab} = \frac{1}{2}\bar{R}\delta^a_a = \bar{R}$. Q.E.D.

(d) [5] Consider the surface of a sphere of radius *r*, with metric

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Compute any non-vanishing component of Riemann and use it to compute the Ricci scalar.

We first use the online routine to get the non-vanishing components of the Christoffel symbols.

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> x:=[theta,phi]:g:=array([[r^2,0],[0,r^2*sin(theta)^2]]):
> Christoffel(g,x);
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$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta.$$

We now grind through a particular component of Riemann, say

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = r^2\left[\partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{e\theta}\Gamma^{e}_{\phi\phi} - \Gamma^{\theta}_{e\phi}\Gamma^{e}_{\phi\theta}\right] \\ &= r^2\left[\partial_\theta\left(-\sin\theta\cos\theta\right) - 0 + 0 - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta}\right] = r^2\left[-\cos^2\theta + \sin^2\theta + \sin\theta\cos\theta\cot\theta\right] \\ &= r^2\sin^2\theta \end{aligned}$$

We then use anti-symmetry to immediately get all the non-vanishing components of the Riemann tensor.

$$R_{\theta\phi\theta\phi} = -R_{\phi\theta\theta\phi} = -R_{\theta\phi\phi\theta} = R_{\phi\theta\phi\theta} = r^2\sin^2\theta \,.$$

We can now deduce $\bar{R}$. Since $R_{abcd} = \frac{1}{2}\bar{R}\varepsilon_{ab}\varepsilon_{cd} = \frac{1}{2}\bar{R}r^4\sin^2\theta[ab][cd]$, we see that $\frac{1}{2}\bar{R}r^4\sin^2\theta = r^2\sin^2\theta$, so $\bar{R} = 2/r^2$. So $R = \bar{R} = 2/r^2$.

20. [10] Work out the components of the four acceleration A^μ in flat space relative to a Lorentz frame defined by

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j$$

where $\partial_t \gamma_{ij} = 0$. Show that its components are given by

$$A^t = \gamma^4 \vec{a} \cdot \vec{v} \quad \text{and} \quad \vec{A} = \gamma^2 \vec{a} + \gamma^4 \left(\vec{a} \cdot \vec{v} \right) \vec{v}$$

where

$$\vec{v} = \frac{d\vec{x}}{dt}, \quad \vec{a} = \frac{d^2 \vec{x}}{dt^2}, \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \vec{v}^2}}.$$

We first need to work out the four velocity. For this we start with the fact that $U_\alpha U^\alpha = -1$, so that

$$-1 = -\left(\frac{dt}{d\tau} \right)^2 + \left(\frac{d\vec{x}}{d\tau} \right)^2 = -\left(\frac{dt}{d\tau} \right)^2 + \left(\frac{dt}{d\tau} \right)^2 \left(\frac{d\vec{x}}{dt} \right)^2 = -\left(\frac{dt}{d\tau} \right)^2 \left(1 - \vec{v}^2 \right)$$

Solving for $dt/d\tau = U^t$, we see that $dt/d\tau = U^t = \gamma = 1/\sqrt{1 - \vec{v}^2}$. The other components are then simply

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \left(\frac{dt}{d\tau} \right) \left(\frac{d\vec{x}}{dt} \right) = \gamma \vec{v}$$

Now, the four-acceleration is defined as $A^\alpha = dU^\alpha / d\tau$. Hence we have

$$A^t = \frac{d}{d\tau} \gamma = \frac{dt}{d\tau} \frac{d\gamma}{dt} = \gamma \left(-\tfrac{1}{2} \right) \left(1 - \vec{v}^2 \right)^{-3/2} \frac{d}{dt} \left(-\vec{v}^2 \right) = \gamma \gamma^3 \left(\vec{v} \cdot \frac{d\vec{v}}{dt} \right) = \gamma^4 \left(\vec{v} \cdot \vec{a} \right),$$

$$\vec{A} = \frac{d}{d\tau} \left(\gamma \vec{v} \right) = \frac{dt}{d\tau} \frac{d}{dt} \left(\gamma \vec{v} \right) = \gamma^2 \frac{d\vec{v}}{dt} + \gamma \vec{v} \frac{d\gamma}{dt} = \gamma^2 \vec{a} + \gamma \vec{v} \gamma^3 \left(\vec{v} \cdot \vec{a} \right) = \gamma^2 \vec{a} + \gamma^4 \left(\vec{v} \cdot \vec{a} \right) \vec{v}$$