Problems 11-15

11. [5] A pion at rest (mass = m_π) decays to a muon (mass = m_μ) and a neutrino (mass = 0). Work out the energy of the muon after the decay.

We let the momentum of the pion be **p**, that of the muon **k**, and that of the neutrino **q**. Then we have $\mathbf{p} = \mathbf{k} + \mathbf{q}$. Now, we want to take the dot product of this with itself, but because we know very little about the neutrino, and are not asked anything about it, it turns out to be easier to rearrange this equation into the form $\mathbf{p} - \mathbf{q} = \mathbf{k}$, and then take the dot product of this with itself:

$$(\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}-\mathbf{q}) = \mathbf{k}\cdot\mathbf{k},$$
$$\mathbf{p}\cdot\mathbf{p} + \mathbf{q}\cdot\mathbf{q} - 2\mathbf{p}\cdot\mathbf{q} = \mathbf{k}\cdot\mathbf{k},$$
$$-m_\pi^2 - m_\mu^2 - 2\mathbf{p}\cdot\mathbf{q} = -m_\nu^2 = 0.$$

To work out the remaining dot product, we note that since the pion is at rest, its four-momentum is $\mathbf{p} = (m_\pi, 0, 0, 0)$ while for the muon, the four-momentum is $\mathbf{q} = (E_\mu, \vec{q})$. The dot product is $\mathbf{p}\cdot\mathbf{q} = -m_\pi E$, so we have

$$-m_\pi^2 - m_\mu^2 + 2m_\pi E = 0,$$
$$E = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}.$$

12. [10] Work out explicitly how the components of the electric and magnetic field mix under (a) a rotation and (b) a Lorentz boost, as given by the following formulas:

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The formula we need to use is $\bar{F}^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu}$. Written as a matrix, this is $\bar{F} = \Lambda F \Lambda^T$, so we have in the first case

$$\bar{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x\cos\theta - E_y\sin\theta & E_x\sin\theta + E_y\cos\theta & E_z \\ -E_x & -B_z\sin\theta & B_z\cos\theta & -B_y \\ -E_y & -B_z\cos\theta & -B_z\sin\theta & B_x \\ -E_z & B_y\cos\theta + B_x\sin\theta & B_y\sin\theta - B_x\cos\theta & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & E_x\cos\theta - E_y\sin\theta & E_x\sin\theta + E_y\cos\theta & E_z \\ -E_x\cos\theta + E_y\sin\theta & 0 & B_z & -B_y\cos\theta - B_x\sin\theta \\ -E_x\sin\theta - E_y\cos\theta & -B_z & 0 & B_x \\ -E_z & B_y\cos\theta + B_x\sin\theta & B_y\sin\theta - B_x\cos\theta & 0 \end{pmatrix}
$$

where we used the identity $\cos^2\theta + \sin^2\theta = 1$ to simplify in the last step. Comparing this with the original form, we see that the transformation is

$$
\begin{aligned}
\bar{E}_x &= E_x\cos\theta - E_y\sin\theta & & \bar{B}_x = B_x\cos\theta - B_y\sin\theta \\
\bar{E}_y &= E_x\sin\theta + E_y\cos\theta & \text{and} \quad & \bar{B}_y = B_x\sin\theta + B_y\cos\theta \\
\bar{E}_z &= E_z & & \bar{B}_z = B_z
\end{aligned}
$$

This is exactly the way we would expect for a pair of vectors. Repeating for the other Lorentz transformation, we see that

$$
\bar{F} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -E_x\sinh\phi & E_x\cosh\phi & E_y & E_z \\ -E_x\cosh\phi & E_x\sinh\phi & B_z & -B_y \\ -E_y\cosh\phi + B_z\sinh\phi & E_y\sinh\phi - B_z\cosh\phi & 0 & B_x \\ -E_z\cosh\phi - B_y\sinh\phi & E_z\sinh\phi + B_y\cosh\phi & -B_x & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & E_x & E_y\cosh\phi - B_z\sinh\phi & E_z\cosh\phi + B_y\sinh\phi \\ -E_x & 0 & B_z\cosh\phi - E_y\sinh\phi & -E_z\sinh\phi - B_y\cosh\phi \\ -E_y\cosh\phi + B_z\sinh\phi & E_y\sinh\phi - B_z\cosh\phi & 0 & B_x \\ -E_z\cosh\phi - B_y\sinh\phi & E_z\sinh\phi + B_y\cosh\phi & -B_x & 0 \end{pmatrix}
$$

This time we find

$$
\begin{aligned}
\bar{E}_x &= E_x & & \bar{B}_x = B_x \\
\bar{E}_y &= E_y\cosh\phi - B_z\sinh\phi & \text{and} \quad & \bar{B}_y = B_y\cosh\phi + E_z\sinh\phi \\
\bar{E}_z &= E_z\cosh\phi + B_y\sinh\phi & & \bar{B}_z = B_z\cosh\phi - E_y\sinh\phi
\end{aligned}
$$

It's interesting that under a Lorentz boost, the electric and magnetic field get mixed together, and in this rather complicated way.

13. [5] Show that $\nabla_\alpha F^{\beta\alpha} = j^\beta / \varepsilon_0$ in Cartesian coordinates automatically implies the conservation of charge $\nabla_\beta j^\beta = 0$.

Taking the derivative of $\nabla_\alpha F^{\beta\alpha}$, we have

$$\nabla_\beta \nabla_\alpha F^{\beta\alpha} = -\nabla_\beta \nabla_\alpha F^{\alpha\beta} = -\nabla_\alpha \nabla_\beta F^{\alpha\beta} = -\nabla_\beta \nabla_\alpha F^{\beta\alpha}.$$

where we used, successively, the fact that F is antisymmetric, the fact that derivatives in flat coordinates commute, and finally that we can relabel indices as we wish. Since $\nabla_\beta \nabla_\alpha F^{\beta\alpha}$ is equal to minus itself, it must vanish, so $\nabla_\beta \nabla_\alpha F^{\beta\alpha} = 0$, which implies $\nabla_\beta j^\beta = 0$.

14. [10] The stress-energy tensor $T^{\alpha\beta}$ is a symmetric tensor, $T^{\alpha\beta} = T^{\beta\alpha}$.

(a) [5] Work out how the components of the rotated tensor $\bar{T}^{\alpha\beta}$ are related to those in the original frame, if we perform a rotation by 90 degrees around the *z*-axis (the first Lorentz transformation in problem 12). You may find it easiest to leave your answer in matrix notation.

We simply perform a rotation like before, and find

$$\bar{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T^{tt} & -T^{ty} & T^{tx} & T^{tz} \\ T^{xt} & -T^{xy} & T^{xx} & T^{xz} \\ T^{yt} & -T^{yy} & T^{yx} & T^{yz} \\ T^{zt} & -T^{zy} & T^{zx} & T^{zz} \end{pmatrix} = \begin{pmatrix} T^{tt} & -T^{ty} & T^{tx} & T^{tz} \\ -T^{yt} & T^{yy} & -T^{yx} & -T^{yz} \\ T^{xt} & -T^{xy} & T^{xx} & T^{xz} \\ T^{zt} & -T^{zy} & T^{zx} & T^{zz} \end{pmatrix}$$

(b) [5] Show that if the stress-energy tensor is unchanged by the rotation in part (a), we must have $T^{tx} = T^{ty} = T^{xy} = T^{xz} = T^{yz} = 0$ and $T^{xx} = T^{yy}$.

Looking at the middle items in the top row, we see that $T^{tx} = T^{ty} = -T^{tx}$, so they both vanish. Similarly, looking at the bottom row, we see that $T^{zx} = T^{zy} = -T^{zx}$, so these vanish, and recalling that T is symmetric, we get two more expressions that vanish. Looking at, say, the second row third column, we see that $T^{xy} = -T^{yx}$, but since T is symmetric, these vanish as well. Finally, matching the middle terms on the diagonal, $T^{xx} = T^{yy}$.

15. [10] Scalar quantities should be agreed on by all observers. Work out the scalar quantities $F_{\alpha\beta}F^{\alpha\beta}$ and $\varepsilon_{\alpha\beta\delta\gamma}F^{\alpha\beta}F^{\delta\gamma}$ in terms of the electric and magnetic fields. Then use the first of these to show that the electromagnetic energy density, $\rho = \frac{1}{2}\varepsilon_0\left(\mathbf{E}^2 + \mathbf{B}^2\right)$ can be written in the form

$$T^{tt} = \varepsilon_0\left(F^t{}_\mu F^{t\mu} - \tfrac{1}{4}\eta^{tt}F^{\nu\mu}F_{\nu\mu}\right).$$

The hardest part of this is simply writing down all the terms. Obviously, we can skip any term that has matching indices on an *F*, which should save us a bit of time. Let's start with $F_{\alpha\beta}F^{\alpha\beta}$, and note that if we swap the two indices we get two minus signs, so we don't need to put in both $F_{xy}F^{xy}$ and $F_{yx}F^{yx}$, for example. So to save time, we'll simply double each term. So we have

$$\begin{aligned} F_{\alpha\beta}F^{\alpha\beta} &= 2\left(F_{tx}F^{tx} + F_{ty}F^{ty} + F_{tz}F^{tz} + F_{xy}F^{xy} + F_{xz}F^{xz} + F_{yz}F^{yz}\right) \\ &= 2\left(-E_x^2 - E_y^2 - E_z^2 + B_z^2 + B_y^2 + B_z^2\right) = 2\left(\vec{B}^2 - \vec{E}^2\right). \end{aligned}$$

Now, for the other expression, we first note that since we are in flat coordinates,

$$\varepsilon_{\alpha\beta\delta\gamma}F^{\alpha\beta}F^{\delta\gamma} = \sqrt{-g}\,[\alpha\beta\gamma\delta]F^{\alpha\beta}F^{\delta\gamma} = [\alpha\beta\gamma\delta]F^{\alpha\beta}F^{\delta\gamma}$$

This expression has twenty-four terms, which complicates things a bit. Now, if we interchange α with β, there is no change in the expression. Similarly, if we interchange the pair $\alpha\beta$ with $\gamma\delta$, there is similarly no change. Between these two choices, we can move any index we want into the first position. One of the indices will be the time index. We'll simply assume $\alpha = t$ and multiply by four to take into account that it might appear in one of the other positions. As for the last two indices, we know we can switch them with impunity. So we'll always choose δ before γ. We need another factor of two from this. Putting it all together, we find

$$\begin{aligned} \varepsilon_{\alpha\beta\delta\gamma}F^{\alpha\beta}F^{\delta\gamma} &= [\alpha\beta\gamma\delta]F^{\alpha\beta}F^{\delta\gamma} = 8\left\{[txyz]F^{tx}F^{yz} + [tyxz]F^{ty}F^{xz} + [tzxy]F^{tz}F^{xy}\right\} \\ &= 8\left\{E_xB_x - E_y\left(-B_y\right) + E_zB_z\right\} = 8\vec{E}\cdot\vec{B} \end{aligned}$$

It remains to compute

$$\begin{aligned} T^{tt} &= \varepsilon_0\left(F^t{}_\mu F^{t\mu} - \tfrac{1}{4}\eta^{tt}F^{\nu\mu}F_{\nu\mu}\right) = \varepsilon_0\left[F^t{}_x F^{tx} + F^t{}_y F^{ty} + F^t{}_z F^{tz} - \tfrac{1}{4}(-1)2\left(\vec{B}^2 - \vec{E}^2\right)\right] \\ &= \varepsilon_0\left[E_x^2 + E_y^2 + E_z^2 + \tfrac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right)\right] = \tfrac{1}{2}\varepsilon_0\left(\vec{E}^2 + \vec{B}^2\right) = \rho \end{aligned}$$