

Solutions to Problems 6-10

6. If we assume $\Gamma_{[jk]}^i = 0$ (the torsion-free condition, show that

$$\Gamma_{jk}^i = \frac{1}{2} g^{i\ell} (\partial_j g_{\ell k} + \partial_k g_{j\ell} - \partial_\ell g_{jk})$$

hint: start with $\nabla_k g_{ij} + \nabla_j g_{ki} - \nabla_i g_{jk} = 0$

he torsion free condition means $\frac{1}{2}(\Gamma_{jk}^i - \Gamma_{kj}^i) = 0$, or $\Gamma_{jk}^i = \Gamma_{kj}^i$. The covariant derivative acting on the metric is assumed to vanish. Writing out the hint explicitly, we have

$$\begin{aligned} 0 &= \nabla_k g_{ij} + \nabla_j g_{ki} - \nabla_i g_{jk} \\ &= \partial_k g_{ij} - g_{mj} \Gamma_{ik}^m - g_{im} \Gamma_{jk}^m + \partial_j g_{ki} - g_{mi} \Gamma_{kj}^m - g_{km} \Gamma_{ij}^m - \partial_i g_{jk} + g_{mk} \Gamma_{ji}^m + g_{jm} \Gamma_{ki}^m \\ &= \partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk} - 2g_{im} \Gamma_{jk}^m, \\ 2g_{im} \Gamma_{jk}^m &= \partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}. \end{aligned}$$

Multiplying both sides by $\frac{1}{2} g^{\ell m}$, we have

$$\begin{aligned} g^{\ell m} g_{im} \Gamma_{jk}^m &= \frac{1}{2} g^{\ell m} (\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}), \\ \Gamma_{jk}^\ell &= \frac{1}{2} g^{\ell m} (\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}). \end{aligned}$$

The last equation is the desired relation once we swap the indices $i \leftrightarrow \ell$.

7. For spherical coordinates,

(a) Determine all eighteen components of Γ_{ij}^k .

The metric and inverse metric are given in the (r, θ, ϕ) basis by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2 \theta \end{pmatrix}.$$

We now simply start working them out, starting with the case $k = r$.

$$\Gamma_{ij}^r = \frac{1}{2} g^{rk} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = \frac{1}{2} (\partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij})$$

The first two terms are non-zero if we pick $i = r$ or $j = r$, but then g has no derivative, so only the last term contributes. So only the last term contributes, and this only occurs if $i = j$ and we don't pick r . So we have

$$\Gamma_{\theta\theta}^r = -\frac{1}{2}\partial_r g_{\theta\theta} = -r, \quad \Gamma_{\phi\phi}^r = -\frac{1}{2}\partial_r g_{\phi\phi} = -r \sin^2 \theta$$

$$\Gamma_{rr}^r = \Gamma_{r\theta}^r = \Gamma_{\theta r}^r = \Gamma_{r\phi}^r = \Gamma_{\phi r}^r = \Gamma_{\theta\phi}^r = \Gamma_{\phi\theta}^r = 0$$

We now move on to $k = \theta$:

$$\Gamma_{ij}^\theta = \frac{1}{2} g^{\theta k} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = \frac{1}{2} r^{-2} (\partial_i g_{j\theta} + \partial_j g_{i\theta} - \partial_\theta g_{ij})$$

The first two terms don't vanish only if one of i or j is θ and the other is r . The last term doesn't vanish only if $i = j = \phi$, so we have

$$\Gamma_{\theta r}^\theta = \Gamma_{r\theta}^\theta = \frac{1}{2} r^{-2} \partial_r g_{\theta\theta} = r^{-1}, \quad \Gamma_{\phi\phi}^\theta = -\frac{1}{2} r^{-2} \partial_\theta g_{\phi\phi} = -\frac{1}{2} \partial_\theta \sin^2 \theta = -\sin \theta \cos \theta,$$

$$\Gamma_{rr}^\theta = \Gamma_{r\phi}^\theta = \Gamma_{\phi r}^\theta = \Gamma_{\theta\theta}^\theta = \Gamma_{\theta\phi}^\theta = \Gamma_{\phi\theta}^\theta = 0$$

Finally, if $k = \phi$, we have

$$\Gamma_{ij}^\phi = \frac{1}{2} g^{\phi k} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) = \frac{1}{2} r^{-2} (\sin \theta)^{-2} (\partial_i g_{j\phi} + \partial_j g_{i\phi} - \partial_\phi g_{ij}).$$

The last term always vanishes, and the other terms don't vanish if one of i or j is ϕ and the other is r or θ , so we have

$$\Gamma_{\phi r}^\phi = \Gamma_{r\phi}^\phi = \frac{1}{2} r^{-2} (\sin \theta)^{-2} \partial_r g_{\phi\phi} = r^{-1}, \quad \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \frac{1}{2} r^{-2} (\sin \theta)^{-2} \partial_\theta g_{\phi\phi} = \cot \theta,$$

$$\Gamma_{rr}^\phi = \Gamma_{r\theta}^\phi = \Gamma_{\theta r}^\phi = \Gamma_{\theta\theta}^\phi = \Gamma_{\phi\phi}^\phi = 0.$$

The non-zero components, therefore, are

$$\Gamma_{\theta\theta}^r = -r, \quad \Gamma_{\phi\phi}^r = -r \sin^2 \theta, \quad \Gamma_{\theta r}^\theta = \Gamma_{r\theta}^\theta = \Gamma_{\phi r}^\phi = \Gamma_{r\phi}^\phi = r^{-1},$$

$$\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \cot \theta.$$

(b) Determine the standard formula for the Laplacian ∇^2 acting on a scalar

The Laplacian can be worked out in a straightforward way, namely

$$\begin{aligned} \nabla^2 \psi &= g^{ij} \nabla_i \nabla_j \psi = g^{ij} [\partial_i (\nabla_j \psi) - \Gamma_{ji}^k \nabla_k \psi] = g^{ij} (\partial_i \partial_j \psi - \Gamma_{ji}^k \partial_k \psi) \\ &= \partial_r^2 \psi - \Gamma_{rr}^k \partial_k \psi + \frac{1}{r^2} (\partial_\theta^2 \psi - \Gamma_{\theta\theta}^k \partial_k \psi) + \frac{1}{r^2 \sin^2 \theta} (\partial_\phi^2 \psi - \Gamma_{\phi\phi}^k \partial_k \psi) \\ &= \partial_r^2 \psi + \frac{1}{r^2} (\partial_\theta^2 \psi + r \partial_r \psi) + \frac{1}{r^2 \sin^2 \theta} (\partial_\phi^2 \psi + r \sin^2 \theta \partial_r \psi + \sin \theta \cos \theta \partial_\theta \psi) \\ &= \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \end{aligned}$$

The equality of the last two lines can be easily checked. It is put into this final “simplified” form for comparison with standard references.

8. Show that $\Gamma_{ki}^i = \partial_k \ln \sqrt{g}$, where g is the determinant of g_{ij} .

This problem starts suspiciously easily:

$$\Gamma_{ki}^i = \frac{1}{2} g^{ij} (\partial_k g_{ji} + \partial_i g_{kj} - \partial_j g_{ki}) = \frac{1}{2} g^{ij} \partial_k g_{ji} \quad (*)$$

where we used the symmetry of the inverse metric to simplify. We now need to work on two closely related things: how one obtains the inverse metric g^{ij} , and how one determines the derivative of the determinant of g .

The inverse metric is found by inverting the matrix g_{ij} , and the easiest way to see this is using Cramer's rule (look it up on the internet, if necessary). Cramer's rule says that the component g^{ij} is found by dividing the *cofactor* of g_{ji} (note the order of the indices has switched) by the determinant of g . The cofactor is defined as the determinant of the *minor* of g_{ji} multiplied by $(-1)^{i+j}$. The *minor* of g_{ji} is the determinant of the matrix g after you remove row j and column i from the matrix. In summary,

$$g^{ij} = (-1)^{i+j} \frac{M_{ji}}{g} \quad (**)$$

Now, we will also need to know how the determinant changes. To find the determinant, we can pick any row i of the matrix, and then the determinant is computed as

$$g = \sum_i g_{ij} (-1)^{i+j} M_{ij}$$

The minor is defined as a determinant itself, so this definition seems circular, but the minor is computed from a matrix that is one size smaller than the original matrix, and therefore this ultimately reduces to a one by one matrix, for which the determinant is simply the number. Note that any particular component of g_{ij} will not appear in its corresponding minor. If we take the derivative of g with respect to a particular component, therefore, we will find

$$\frac{\partial g}{\partial g_{ij}} = (-1)^{i+j} M_{ij}$$

We can now compute the derivative of the determinant of g , using the chain rule:

$$\frac{\partial g}{\partial x^k} = \frac{\partial g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial x^k} = (-1)^{i+j} M_{ij} \frac{\partial g_{ij}}{\partial x^k}$$

Exchanging the roles of the dummy indices i and j , and comparison with $(**)$ then tells us

$$\partial_k g = g g^{ij} \partial_k g_{ji}$$

Substitution in $(*)$ then tells us

$$\Gamma_{ki}^i = \frac{1}{2} g^{ij} \partial_k g_{ji} = \frac{\partial_k g}{2g} = \frac{1}{2} \partial_k \ln g = \partial_k \ln \sqrt{g}.$$

9. For cylindrical coordinates with metric $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$

(a) Find an orthonormal basis of vectors and forms. To save your instructor from insanity, use the most obvious choice.

The metric and inverse metric in the basis (ρ, ϕ, z) is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We want to write our metric in the form $ds^2 = \delta_{ij} \tilde{\omega}^i \tilde{\omega}^j$. This can clearly be done if

$$\tilde{\omega}^{\hat{\rho}} = d\rho, \quad \tilde{\omega}^{\hat{\phi}} = \rho d\phi, \quad \tilde{\omega}^{\hat{z}} = dz.$$

We then want to pick a dual basis of vectors \mathbf{e}_i with the property $\langle \tilde{\omega}^i, \mathbf{e}_j \rangle = \delta_j^i$, which is easy to figure out:

$$\mathbf{e}_{\hat{\rho}} = \partial_{\rho}, \quad \mathbf{e}_{\hat{\phi}} = \frac{1}{\rho} \partial_{\phi}, \quad \mathbf{e}_{\hat{z}} = \partial_z.$$

(b) Find ∇f , $\nabla \cdot \mathbf{v}$ and $\nabla^2 f$ in a coordinate basis

In any coordinates, $\nabla f = g^{ij} \partial_j f \partial_i$. For the divergence, we use the formula found in class, $\nabla \cdot \mathbf{v} = \partial_i (\sqrt{g} v^i) / \sqrt{g}$. For the Laplacian, $\nabla^2 f = \partial_i (\sqrt{g} g^{ij} \partial_j f) / \sqrt{g}$. It is easy to see that $\sqrt{g} = \sqrt{\rho^2} = \rho$, so we have

$$\nabla f = \partial_{\rho} f \partial_{\rho} + \frac{1}{\rho^2} \partial_{\phi} f \partial_{\phi} + \partial_z f \partial_z,$$

$$\nabla \cdot \mathbf{v} = \frac{1}{\rho} \left[\partial_{\rho} (\rho v^{\rho}) + \partial_{\phi} (\rho v^{\phi}) + \partial_z (\rho v^z) \right] = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v^{\rho}) + \frac{\partial v^{\phi}}{\partial \phi} + \frac{\partial v^z}{\partial z},$$

$$\nabla^2 f = \partial_{\rho} (\rho \partial_{\rho} f) / \rho + \partial_{\phi} (\rho \rho^{-2} \partial_{\phi} f) / \rho + \partial_z (\rho \partial_z f) / \rho = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

(c) Find ∇f , $\nabla \cdot \mathbf{v}$ and $\nabla^2 f$ in the orthonormal basis.

The gradient is computed merely by substituting the appropriate expressions for the coordinate vectors in terms of the orthonormal vectors. The divergence is identical, except we have to write the coordinate basis in terms of the orthonormal basis by writing $v^i = e_{\hat{a}}^i v^{\hat{a}}$, where $e_{\hat{a}}^i = e_{\hat{a}}^i \partial_i$. This makes it easy to see that $v^{\rho} = v^{\hat{\rho}}$ and $v^z = v^{\hat{z}}$, but $v^{\phi} = v^{\hat{\phi}} / \rho$. For the Laplacian, everything is unchanged. So we have

$$\begin{aligned}\nabla f &= \mathbf{e}_\rho \partial_\rho f + \mathbf{e}_\phi \frac{1}{\rho} \partial_\phi f + \mathbf{e}_z \partial_z f, \\ \nabla \cdot \mathbf{v} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v^\rho) + \frac{1}{\rho} \frac{\partial v^\phi}{\partial \phi} + \frac{\partial v^z}{\partial z}, \\ \nabla^2 f &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.\end{aligned}$$

10. A rocket follows a path with coordinates given by

$$\begin{aligned}x &= \sqrt{t^2 + a^{-2}} - a^{-1}, \\ y &= z = 0.\end{aligned}$$

(a) Write all four coordinates in terms of the proper time τ .

$$\tau = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \int \sqrt{dt^2 - \left(\frac{tdt}{\sqrt{t^2 + a^{-2}}} \right)^2} = \int dt \sqrt{\frac{t^2 + a^{-2} - t^2}{t^2 + a^{-2}}} = \int \frac{dt}{\sqrt{a^2 t^2 + 1}}.$$

This integral can be attacked in a variety of ways, but the quickest way to finish it is to substitute $t = a^{-1} \sinh \phi$, and use the identities $d \sinh \phi = \cosh \phi d\phi$ and $\sinh^2 \phi + 1 = \cosh^2 \phi$. So we have

$$\tau = \int \frac{a^{-1} d \sinh \phi}{\sqrt{a^2 a^{-2} \sinh^2 \phi + 1}} = \frac{1}{a} \int \frac{\cosh \phi d\phi}{\cosh \phi} = \frac{\phi}{a}$$

Substituting this in, we have

$$\begin{aligned}t &= a^{-1} \sinh \phi = a^{-1} \sinh(a\tau), \\ x &= \sqrt{a^{-2} \sinh^2(a\tau) + a^{-2}} - a^{-1} = a^{-1} \sqrt{\sinh^2(a\tau) + 1} - a^{-1} = a^{-1} [\cosh(a\tau) - 1]\end{aligned}$$

and of course, $y = z = 0$.

(b) Show that the proper acceleration is given by $\sqrt{\mathbf{A}^2} = a$.

The proper acceleration vector is

$$\begin{aligned}A^\mu &= \frac{d^2}{d\tau^2} (a^{-1} \sinh(a\tau), a^{-1} [\cosh(a\tau) - 1], 0, 0) = \frac{d^2}{d\tau^2} (\cosh(a\tau), \sinh(a\tau), 0, 0) \\ &= (a \sinh(a\tau), a \cosh(a\tau), 0, 0).\end{aligned}$$

Its magnitude is

$$\sqrt{\mathbf{A}^2} = \sqrt{-a^2 \sinh^2(a\tau) + a^2 \cosh^2(a\tau)} = \sqrt{a^2} = a.$$

- (c) Calculate the proper time required to reach α Centauri (4.3 ly), the center of the Milky Way galaxy (25 kly) and the edge of the universe (40 Gly), if accelerating at a constant proper acceleration $a = 9.80 \text{ m/s}^2$. A light-year (ly) is the speed of light c times one year.

We first need to convert everything into the same units. The distances are all easily rewritten as times, so let's convert the acceleration into time as well, and use units of years everywhere. We have

$$a = 9.80 \text{ m/s}^2 = \frac{(9.80 \text{ m/s}^2)(3.156 \times 10^7 \text{ s/y})}{(3.00 \times 10^8 \text{ m/s})} = 1.031 \text{ y}^{-1}$$

So far so good! We now work out the proper time by solving the relation between x and τ for τ .

$$\begin{aligned} x &= a^{-1} [\cosh(a\tau) - 1], \\ ax + 1 &= \cosh(a\tau) \\ \tau &= \frac{\cosh^{-1}(ax + 1)}{a} = \frac{\cosh^{-1}[1.031x + 1]}{1.031} \text{ y} \end{aligned}$$

where the last line assumes x is in years (which is the same as light-years). Most good calculators have the inverse hyperbolic cosine built in, and we find

$$\begin{aligned} \tau_{\alpha\text{Cent}} &= \cosh^{-1}(1.031 \times 4.3 + 1) / 1.031 = 2.31 \text{ y}^{-1} \\ \tau_{\text{MW}} &= \cosh^{-1}(1.031 \times 2.5 \times 10^4 + 1) / 1.031 = 10.5 \text{ y}^{-1} \\ \tau_{\text{Uni}} &= \cosh^{-1}(1.031 \times 4.0 \times 10^{10} + 1) / 1.031 = 24.4 \text{ y}^{-1} \end{aligned}$$

These times are all so long that you could do it, but they would eat up substantial chunks of your life! In actuality, most of them would take longer since you would want to stop once you get there.