Solutions to Problems 48-50

48. Show that $\nabla_a \perp_b^c = -s_b K_a{}^c - s_a s_b a^c - s^c K_{ab} - s^c s_a a_b$ **.**

From the notes, we know that $\varepsilon \nabla_a s_b = K_{ab} + s_a a_b$, so that

$$\nabla_a \perp_b^c = \nabla_a \left(\delta_b^c - \varepsilon s^c s_b \right) = -\varepsilon \left(\nabla_a s^c \right) s_b - \varepsilon s^c \nabla_a s_b = -\left(K_a{}^c + s_a a^c \right) s_b - s^c \left(K_{ab} + s_a a_b \right)$$
$$= -s_b K_a{}^c - s_a s_b a^c - s^c K_{ab} - s^c s_a a_b .$$

49. Show that $\perp \nabla_a \nabla_b s_c = \varepsilon D_a K_{bc} + a_c K_{ab}$ **.**

Recalling that $\perp \nabla_a T \equiv D_a T$ for any tensor, we have

$$\perp \nabla_a \nabla_b s_c = D_a \left(\nabla_b s_c \right) = \varepsilon D_a \left(K_{bc} + s_b a_c \right) = \varepsilon D_a K_{bc} + \varepsilon \perp \nabla_a \left(s_b a_c \right)$$
$$= \varepsilon D_a K_{bc} + \varepsilon \perp s_b \nabla_a a_c + \perp \varepsilon a_c \nabla_a s_b$$
$$= \varepsilon D_a K_{bc} + 0 + \perp a_c \left(K_{ab} + s_a a_b \right) = \varepsilon D_a K_{bc} + \perp a_c K_{ab}$$

We almost have it! The $\perp$ has no effect on the extrinsic curvature, because it is automatically perpendicular to the normal. On the acceleration, we have

$$\perp a_c = a_c - \varepsilon s^d a_d s_c = a_c - \varepsilon s_c s^d s^e \nabla_e s_d = a_c - \tfrac{1}{2} \varepsilon s_c s^e \nabla_e \left(s_d s^d \right) = a_c - \tfrac{1}{2} \varepsilon s_c s^e \nabla_e \varepsilon = a_c$$

and therefore we have $\perp \nabla_a \nabla_b s_c = \varepsilon D_a K_{bc} + a_c K_{ab}$.

50. Show that $s^a \mathcal{L}_{\vec{s}} K_{ab} = 0$ **and therefore** $\perp \mathcal{L}_{\vec{s}} K_{ab} = \mathcal{L}_{\vec{s}} K_{ab}$ **.**

Writing out the Lie derivative explicitly, we have

$$s^a \mathcal{L}_{\vec{s}} K_{ab} = s^a \left(s^c \nabla_c K_{ab} + K_{cb} \nabla_a s^c + K_{ac} \nabla_b s^c \right)$$
$$= s^a s^c \nabla_c K_{ab} + s^a \left[\nabla_a \left(K_{cb} s^c \right) - s^c \nabla_a K_{cb} \right] + 0 = 2 s^a s^c \nabla_{[c} K_{a]b} = 0$$

We used the fact that K is orthogonal to s, and in the last step, we have a symmetric tensor times an anti-symmetric one. It follows that

$$\perp \mathcal{L}_{\vec{s}} K_{ab} = \left(\delta_a^c - \varepsilon s_a s^c \right) \left(\delta_b^d - \varepsilon s_b s^d \right) \mathcal{L}_{\vec{s}} K_{cd} = \delta_a^c \delta_b^d \mathcal{L}_{\vec{s}} K_{cd} = \mathcal{L}_{\vec{s}} K_{ab}$$