Solutions to Problems 1-5

1. Define a dot product for 1-forms $\tilde{\omega}^i \cdot \tilde{\omega}^j \equiv g^{ij}$

(a) Show that $\tilde{u} \cdot \tilde{v} \equiv \langle \tilde{u}, \vec{v} \rangle$ implies $u^i = g^{ij} u_j$

This problem is a bit confusing as worded, since in the "definition" $\tilde{\omega}^i \cdot \tilde{\omega}^j \equiv g^{ij}$, neither side is defined. In part (a), we can treat $\tilde{u} \cdot \tilde{v} \equiv \langle \tilde{u}, \vec{v} \rangle$ as a definition, in the sense that if a vector $\vec{v}$ has a corresponding 1-form (which I would call a covector) $\tilde{v}$, this allows us to define a dot product between another 1-form and that 1-form.

Now we can try to figure out what the "definition" $\tilde{\omega}^i \cdot \tilde{\omega}^j \equiv g^{ij}$ means. First, it may be that there is a vector $\vec{\omega}_j$ such that when it is turned into a 1-form, it yields $\tilde{\omega}^j$. Then we treat the first statement as the definition of g^{ij}.

Now, we want to show that $u^i = g^{ij} u_j$ must follow from the previous two definitions. To show this, we simply start with the right hand side of this equation, and try to turn it into the left-hand side of the equation.

$$g^{ij} u_j = \tilde{\omega}^i \cdot \tilde{\omega}^j u_j = \tilde{\omega}^i \cdot \tilde{u} = \langle \tilde{\omega}^i, \vec{u} \rangle = \langle \tilde{\omega}^i, u^j \vec{e}_j \rangle = \delta^i_j u^j = u^i$$

That wasn't too bad!

(b) Show that if we think of g_{ij} as a matrix, then the matrix $g^{ij} = (g_{ij})^{-1}$.

hint: $M^{-1} \cdot M = \mathbf{1}$

To prove this, consider an arbitrary vector with components u^i. We know that $u_i = g_{ij} u^j$ and $u^i = g^{ij} u_j$, so we have

$$u^i = g^{ij} u_j = g^{ij} g_{jk} u^k$$

Now, it is also obvious that $u^i = \delta^i_k u^k$, so we have for any vector

$$g^{ij} g_{jk} u^k = \delta^i_k u^k$$

The only way this can be true for any vector is if $g^{ij} g_{jk} = \delta^i_k$, which in matrix notation means g_{ij} and g^{ij} are inverses of each other.

2. Consider a 2d skew Cartesian coordinates, related to conventional coordinates by

$$x = \bar{x} + \bar{y}\sin\theta$$
$$y = \bar{y}\cos\theta$$

(a) Express $d\bar{x}^i$ in terms of dx^i.

This is most easily done simply by letting the differential act on each equation, which yields

$$dx = d\bar{x} + d\bar{y}\sin\theta$$
$$dy = d\bar{y}\cos\theta$$

Since θ is a constant, the differential doesn't affect them. Solving these for the other side of the equation, we have

$$d\bar{x} = dx - dy\tan\theta$$
$$d\bar{y} = dy\sec\theta$$

(b) Express $\partial_{\bar{x}^i}$ in terms of ∂_{x^i}

This is most easily found by thinking of them as partial derivatives:

$$\partial_{\bar{x}} = \frac{\partial}{\partial\bar{x}} = \frac{\partial x}{\partial\bar{x}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\bar{x}}\frac{\partial}{\partial y} = \partial_x,$$
$$\partial_{\bar{y}} = \frac{\partial}{\partial\bar{y}} = \frac{\partial x}{\partial\bar{y}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\bar{y}}\frac{\partial}{\partial y} = \sin\theta\partial_x + \cos\theta\partial_y.$$

(c) Find the components of g_{ij} and g^{ij}.

We know the metric in Cartesian coordinates; it is

$$ds^2 = \left(d\bar{x} + d\bar{y}\sin\theta\right)^2 + \left(d\bar{y}\cos\theta\right)^2 = d\bar{x}^2 + 2d\bar{x}d\bar{y}\sin\theta + d\bar{y}^2$$

This lets us pick out the coefficients of the metric. Keeping in mind that we must divide by two for the off-diagonal elements, we have

$$g_{\bar{x}\bar{x}} = 1, \quad g_{\bar{x}\bar{y}} = g_{\bar{y}\bar{x}} = \sin\theta, \quad g_{\bar{y}\bar{y}} = 1, \quad \text{or} \quad g_{ij} = \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix}.$$

This can be written as a two by two matrix. The inverse matrix can be found by a variety of techniques. The result is

$$g^{ij} = \frac{1}{1-\sin^2\theta}\begin{pmatrix} 1 & -\sin\theta \\ -\sin\theta & 1 \end{pmatrix} = \begin{pmatrix} \sec^2\theta & -\sec\theta\tan\theta \\ -\sec\theta\tan\theta & \sec^2\theta \end{pmatrix}.$$

(d) Find the lengths of $\partial_{\bar{x}}$ and $\partial_{\bar{y}}$ and the angle between them.

$$\left|\partial_{\bar{x}}\right| = \sqrt{\partial_{\bar{x}} \cdot \partial_{\bar{x}}} = \sqrt{\partial_x \cdot \partial_x} = 1$$

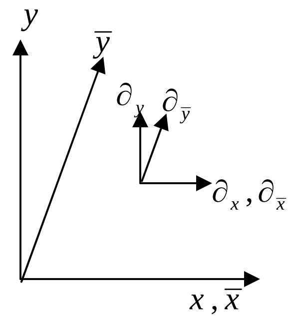
$$\left|\partial_{\bar{y}}\right| = \sqrt{\partial_{\bar{y}} \cdot \partial_{\bar{y}}} = \sqrt{\left(\sin\theta \partial_x + \cos\theta \partial_y\right) \cdot \left(\sin\theta \partial_x + \cos\theta \partial_y\right)} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$\cos\alpha = \frac{\partial_{\bar{x}} \cdot \partial_{\bar{y}}}{\left|\partial_{\bar{x}}\right|\left|\partial_{\bar{y}}\right|} = \frac{\partial_x \cdot \left(\sin\theta \partial_x + \cos\theta \partial_y\right)}{1} = \sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right),$$

$$\alpha = \tfrac{\pi}{2} - \theta$$

(e) Draw the vectors ∂_x, ∂_y, $\partial_{\bar{x}}$ and $\partial_{\bar{y}}$ on a diagram of the coordinate system at one point.

The diagram appears at right. The vectors requested are parallel to the coordinate axes, and all have unit length.



3. Spherical coordinates are defined by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

(a) Show that $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

The appropriate differentials are simply

$$dx = \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi,$$
$$dy = \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi,$$
$$dz = \cos\theta dr - r\sin\theta d\theta.$$

Now we take the 3D Cartesian metric formula and substitute in all these differentials:

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= \left[\cos\phi\left(\sin\theta dr + r\cos\theta d\theta\right) - r\sin\theta\sin\phi d\phi\right]^2 + \\
&\left[\sin\phi\left(\sin\theta dr + r\cos\theta d\theta\right) + r\sin\theta\cos\phi d\phi\right]^2 + \left(\cos\theta dr - r\sin\theta d\theta\right)^2 \\
&= \cos^2\phi\left(\sin\theta dr + r\cos\theta d\theta\right)^2 - 2r\sin\theta\sin\phi\cos\phi\left(\sin\theta dr + r\cos\theta d\theta\right)d\phi \\
&+ r^2\sin^2\theta\sin^2\phi d\phi^2 + \sin^2\phi\left(\sin\theta dr + r\cos\theta d\theta\right)^2 \\
&+ 2r\sin\theta\sin\phi\cos\phi\left(\sin\theta dr + r\cos\theta d\theta\right)d\phi + r^2\sin^2\theta\cos^2\phi d\phi^2 \\
&+ \left(\cos\theta dr - r\sin\theta d\theta\right)^2 \\
&= \sin^2\theta dr^2 + 2r\sin\theta\cos\theta dr d\theta + r^2\cos^2\theta d\theta^2 + r^2\sin^2\theta d\phi^2 \\
&+ \cos^2\theta dr^2 - 2r\sin\theta\cos\theta dr d\theta + r^2\sin^2\theta d\theta^2 \\
&= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2
\end{aligned}$$

The metric is then just

$$g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta, \quad g_{r\theta} = g_{\theta r} = g_{r\phi} = g_{\phi r} = g_{\theta\phi} = g_{\phi\theta} = 0.$$

Or, written as a matrix, as given in the problem.

(b) What is the volume element d^3V in spherical coordinates?

The volume element is

$$d^3V = dr\, d\theta\, d\phi\sqrt{\det\left(g_{ij}\right)} = r^2\sin\theta dr\, d\theta\, d\phi.$$

4. Show that $\nabla_i \alpha_j = \partial_i \alpha_j - \Gamma^k_{ji}\alpha_k$. Use Leibnitz's rule, $\nabla_i(AB) = (\nabla_i A)B + A\nabla_i B$.

As discussed in class, we wish to have $\nabla_i(\alpha_j v^j) = \partial_i(\alpha_j v^j)$. We therefore have

$$\begin{aligned}
\partial_i(\alpha_j v^j) &= \nabla_i(\alpha_j v^j) \\
(\partial_i \alpha_j)v^j + \alpha_j \partial_i v^j &= (\nabla_i \alpha_j)v^j + \alpha_j \nabla_i v^j = (\nabla_i \alpha_j)v^j + \alpha_j \partial_i v^j + \alpha_j \Gamma^j_{ki} v^k, \\
(\nabla_i \alpha_j)v^j &= (\partial_i \alpha_j)v^j - (\alpha_j \Gamma^j_{ki})v^k = (\partial_i \alpha_j - \alpha_k \Gamma^k_{ji})v^j.
\end{aligned}$$

In the last step, note that *j* and *k* are "dummy indices," indices that are summed over and therefore we can change their name with impunity. Hence we simply swapped the names $j \leftrightarrow k$ in the last term. The only way this can be true for any vector field v^j is if the coefficients match, so that $\nabla_i \alpha_j = \partial_i \alpha_j - \alpha_k \Gamma^k_{ji}$, the desired relationship.

5. Use the fact that $\nabla_i v^j$ transforms as a tensor to prove one of the following two relationships:

$$\bar{\Gamma}^i_{jk} = \Gamma^a_{bc} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial \bar{x}^i}{\partial x^\ell} \frac{\partial^2 x^\ell}{\partial \bar{x}^j \partial \bar{x}^k} = \Gamma^a_{bc} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k} - \frac{\partial x^\ell}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial^2 \bar{x}^i}{\partial x^\ell \partial x^m}.$$

The fact that this expression is a tensor means that

$$\bar{\nabla}_i \bar{v}^j = \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \nabla_a v^b$$

Expanding out each of the covariant derivatives, we find

$$\partial_{\bar{i}} \bar{v}^j + \bar{v}^k \bar{\Gamma}^j_{ki} = \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \left(\partial_a v^b + v^c \Gamma^b_{ca} \right)$$

Since *v* is a vector, we already know that

$$\bar{v}^j = \frac{\partial \bar{x}^j}{\partial x^a} v^a$$

We also know that

$$\partial_{\bar{i}} = \frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial}{\partial x^a}$$

We therefore have

$$\begin{aligned}
\frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial}{\partial x^a} \left(\frac{\partial \bar{x}^j}{\partial x^b} v^b \right) + \frac{\partial \bar{x}^k}{\partial x^a} v^a \bar{\Gamma}^j_{ki} &= \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \left(\partial_a v^b + v^c \Gamma^b_{ca} \right), \\
\frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^a \partial x^b} v^b + \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial v^b}{\partial x^a} + \frac{\partial \bar{x}^k}{\partial x^a} v^a \bar{\Gamma}^j_{ki} &= \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial v^b}{\partial x^a} + \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ca} v^c,
\end{aligned}$$

$$\frac{\partial \bar{x}^k}{\partial x^a} v^a \bar{\Gamma}^j_{ki} = \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ca} v^c - \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^a \partial x^b} v^b .$$

To understand the meaning of this formula, change the indices so that v always has the same index.

$$\frac{\partial \bar{x}^k}{\partial x^a} \bar{\Gamma}^j_{ki} v^a = \frac{\partial x^c}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ac} v^a - \frac{\partial x^b}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^b \partial x^a} v^a$$

This is only true if the coefficients of v^a match on both sides of the equation, so

$$\frac{\partial \bar{x}^k}{\partial x^a} \bar{\Gamma}^j_{ki} = \frac{\partial x^c}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ac} - \frac{\partial x^b}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^b \partial x^a}$$

We need to solve for $\bar{\Gamma}^j_{ki}$, which can be done by multiplying by the inverse matrix of $\partial \bar{x}^k / \partial x^a$. To do so, multiply by $\partial x^a / \partial \bar{x}^\ell$. So we have

$$\frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial \bar{x}^k}{\partial x^a} \bar{\Gamma}^j_{ki} = \frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial x^c}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ac} - \frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial x^b}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^b \partial x^a},$$

$$\delta^k_\ell \bar{\Gamma}^j_{ki} = \frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial x^c}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ac} - \frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial x^b}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^b \partial x^a},$$

$$\bar{\Gamma}^j_{\ell i} = \frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial x^c}{\partial \bar{x}^i} \frac{\partial \bar{x}^j}{\partial x^b} \Gamma^b_{ac} - \frac{\partial x^a}{\partial \bar{x}^\ell} \frac{\partial x^b}{\partial \bar{x}^i} \frac{\partial^2 \bar{x}^j}{\partial x^b \partial x^a}.$$

The final expression is identical to one of the ones we were asked to prove.