

## Problems 37-40

37. Although it is likely that the “dark energy” component of the universe is a vacuum contribution with  $P/\rho = -1$ , other possibilities can be considered. For example, suppose that  $P/\rho = w$ , where  $w$  is a constant slightly smaller than  $-1$ . A formula for the dependence of the energy density  $\rho$  in terms of the scale factor  $a$  was given in class.

- (a) Assume a flat universe with all matter in the form of the dark energy. Solve the Friedmann equation  $\dot{a}^2/a^2 = \frac{8}{3}\pi G\rho$  for  $a$  as a function of time. You may choose your zero of time and overall scale  $a$  as convenient.
- (b) Show that if  $w < -1$ , the universe will expand to infinite size in finite time (the “Big Rip”). Find a formula for that time, compared to now, based on the current density of dark energy  $\rho_0$  and the state parameter  $w$ .
- (c) Experimentally, the current density of dark energy is about  $\frac{8}{3}\pi G\rho = \Omega_\Lambda H_0^2$ , where  $\Omega_\Lambda \approx 0.74$ ,  $H_0^{-1} \approx 13.4 \text{ Gyr}$ , and  $w = -0.97 \pm 0.07$ . Assuming  $w$  is within one error bar of its actual value, what is the soonest the “Big Rip” could occur? (Note: This answer will be slightly off because the universe is not yet completely dominated by dark energy).

38. Show that the difference in the Riemann tensor between a reference metric  $\hat{g}_{\mu\nu}$  and the actual metric  $g_{\mu\nu}$  is given by

$$\delta R^\ell_{kji} \equiv R^\ell_{kji} - \hat{R}^\ell_{kji} = \hat{\nabla}_j \delta \Gamma^\ell_{ki} - \hat{\nabla}_i \delta \Gamma^\ell_{kj} + \delta \Gamma^m_{ki} \delta \Gamma^\ell_{mj} - \delta \Gamma^m_{kj} \delta \Gamma^\ell_{mi}$$

Then show the trivial consequence

$$\delta R_{ki} \equiv R_{ki} - \hat{R}^j_{kji} = \hat{\nabla}_j \delta \Gamma^j_{ki} - \hat{\nabla}_i \delta \Gamma^j_{kj} + \delta \Gamma^m_{ki} \delta \Gamma^j_{mj} - \delta \Gamma^m_{kj} \delta \Gamma^j_{mi}$$

39. Suppose we have a metric  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , where  $\eta_{\mu\nu}$  is flat spacetime (in arbitrary coordinates), where  $h_{\mu\nu}$  is small. Show that

$$R_{\mu\nu} = -\frac{1}{2} \hat{\nabla}_\rho \hat{\nabla}^\rho h_{\mu\nu} + \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^\rho h_{\rho\nu} + \frac{1}{2} \hat{\nabla}_\nu \hat{\nabla}^\rho h_{\rho\mu} - \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}_\nu (\eta^{\rho\sigma} h_{\rho\sigma}) + O(h^2)$$

where the covariant derivatives are with respect to the flat metric, and all indices are raised and lowered with respect to the flat metric.

40. Under the same conditions as problem 39, show that

$$G_{\mu\nu} = \frac{1}{2} \left\{ \hat{\nabla}^\rho \hat{\nabla}_\mu \bar{h}_{\rho\nu} + \hat{\nabla}^\rho \hat{\nabla}_\nu \bar{h}_{\rho\mu} - \eta_{\mu\nu} \hat{\nabla}^\rho \hat{\nabla}^\sigma \bar{h}_{\rho\sigma} - \hat{\nabla}^\rho \hat{\nabla}_\rho \bar{h}_{\mu\nu} \right\} + O(h^2)$$

where  $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} h_{\rho\sigma}$ .