Problems 29-31

29. An object is dropped from rest at infinity directly into a Schwarzschild metric, which we will assume is valid all the way to $r = 0$.
    (a) What is the initial "energy" E and "angular momentum" L? Calculate a formula for $u^r = dr/d\tau$ as a function of r. Also, find a formula for dr/dt.
    (b) Integrate the equation you found in (a) to find the proper time τ it takes to fall from arbitrary r to the origin.
    (c) Show that the coordinate time t it takes to fall from arbitrary r to just the "event horizon" at $2GM$ is, in contrast, infinite.

30. An incautious traveler has just passed the Schwarzschild radius, so he is now just inside $r = 2GM$ and moving inwards, $u^r < 0$. Using the fact that $u^\alpha u_\alpha = -1$, show that even if he is allowed to accelerate, he can never stop falling in (*i.e.*, he can't have $u^r = 0$), that the magnitude of $|u^r|$ will have a minimum value as a function of r, and find the maximum proper time before his world line terminates, *i.e.*, he reaches $r = 0$.

31. Although four-velocity doesn't exactly apply to photons, we can define an affine parameter λ along the photon's path, and then define $u^\alpha = dx^\alpha/d\lambda$. The geodesic equation for u^α is the same as usual $du^\alpha/d\lambda = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu$, and therefore in the Schwarzschild metric, $u_t = -E$ and $u_\phi = L$ will still be conserved. Our goal in this problem is to find the cross-section for a photon to be absorbed by a black hole.
    (a) Use the fact that $u^\alpha u_\alpha = 0$ for photons to find a formula for u^r for a photon as a function of r, E, L, and M.
    (b) The formula you just found should have $\left(u^r\right)^2 = \infty$ at $r = 0$, then it should falls for a while, and then rises again to its ultimate value at $r = \infty$. It therefore has a global minimum somewhere in between. Find the value of r where this occurs, and find the value of $\left(u^r\right)^2$ there, as a function of L, E, and M.
    (c) If the value you found in part (b) is positive, then u^r never vanishes, which means the photon continues all the way to the singularity at $r = 0$. If the value you found is negative, then it must have been zero somewhere, and therefore the photon must have turned around and left again. For what values of L is the photon absorbed?
    (d) A photon comes in from infinity, such that it has an impact parameter of b; that is, were it not for gravity, it would miss the black hole by a distance b. What is the quantity L for this photon in terms of b and E? (this can be calculated far away, when gravity is negligible)? Find the cross section of the black hole for photons.